Game Theory, Spring 2024

Lecture # 7

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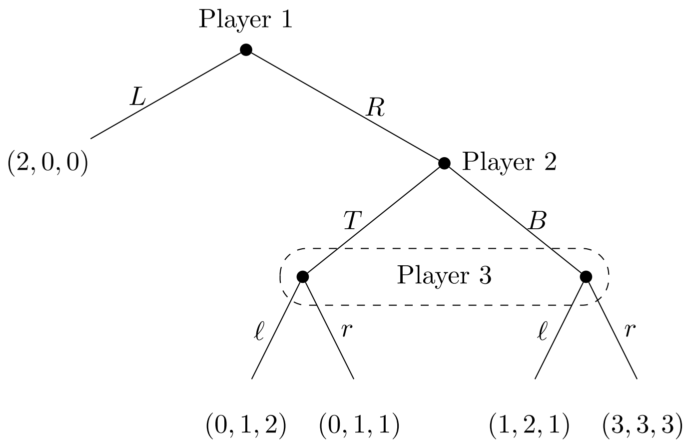
1 Sequential equilibria

1.1 Motivating example

Let us revisit Example 3 from Lecture #5:

Example 3. *Consider the following extensive-form game:*



Recall that we have established that (R, B, r) is the unique subgame-perfect Nash equilibrium in Example 3, but both $\big((R, B, r); \mu^* = 0\big)$ and $\big((L, B, \ell); \mu^* \in [\frac{2}{3}, 1]\big)$ are weak perfect Bayesian equilibria. We have argued informally that the beliefs in $\big((L, B, \ell); \mu^* \in [\frac{2}{3}, 1]\big)$ are not particularly plausible because player 3 should expect player 2 to play B with probability one. Bayes' rule however does not help us to rule out these equilibria because it does not impose any restrictions on beliefs at information sets reached with probability zero. To assign plausible beliefs to information sets reached with probability zero, we consider sequences of completely mixed strategies. In Example 3, we introduce the following:

- Suppose there is $\{p^{(n)}\}_{n=1}^{+\infty}$ such that $0 < p^{(n)} < 1$ for every n, and player 1 plays $p^{(n)}L + (1-p^{(n)})R$ and moreover $\lim_{n\to+\infty} p^n = 1$.
- Suppose there is $\{q^{(n)}\}_{n=1}^{+\infty}$ such that $0 < q^{(n)} < 1$ for every n, and player 2 plays $q^{(n)}T + (1-q^{(n)})B$ and moreover $\lim_{n\to+\infty} q^n = 0$.

We can now define a sequence of beliefs $\{\mu^{(n)}\}_{n=1}^{\infty}$, where $\mu^{(n)}$ the belief of player 3 that she is at history RT, and is derived via Bayes' rule for every n:

$$\mu^{(n)} = \frac{\left(1-p^{(n)}\right)q^{(n)}}{\left(1-p^{(n)}\right)q^{(n)} + \left(1-p^{(n)}\right)\left(1-q^{(n)}\right)} = q^{(n)}.$$

Observe that $\lim_{n\to+\infty} \mu^{(n)} = \lim_{n\to+\infty} q^{(n)} = 0 \notin [\frac{2}{3}, 1]$.

1.2 Sequential equilibrium: definition

We generalize the above approach by introducing the notion of *consistent beliefs*:

Definition 1 (Consistency). *A belief system μ is consistent with a strategy profile σ if there is a sequence of completely mixed behavior strategy profiles $\{\sigma^{(n)}\}_{n=1}^{+\infty}$ such that*

1. $\lim_{n\to\infty} \sigma^{(n)} = \sigma$,
2. $\lim_{n\to\infty} \mu^{(n)} = \mu$, *where $\mu^{(n)}$ is derived from $\sigma^{(n)}$ via Bayes' rule for every n.*

Sequential equilibrium is defined as follows:

Definition 2. *A strategy profile σ and a belief system μ is a sequential equilibrium if*

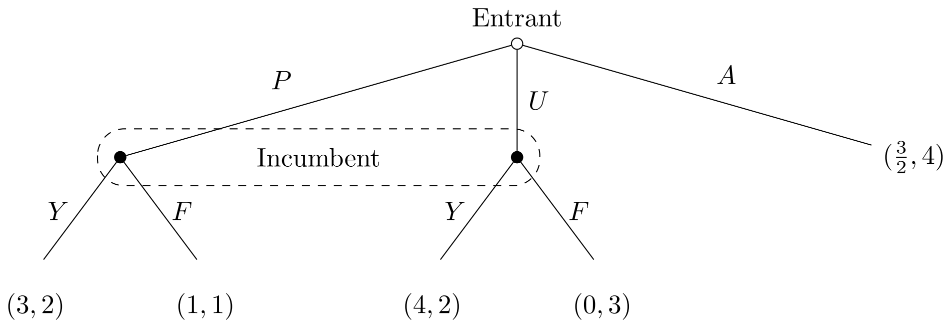
1. *σ is sequentially rational given μ,*
2. *μ is consistent with σ.*

In Example 3, none of $\left((L,B,\ell);\mu^* \in [\frac{2}{3},1]\right)$ is a sequential equilibrium since all beliefs in $[\frac{2}{3},1]$ are inconsistent with (L,B,ℓ).

1.3 More examples

Let us revisit Example 5 from Lecture #6:

Example 5. *Consider the following extensive-form game:*



Recall that we have established that $\big((A, F); \mu^* \in [0, \frac{1}{2}]\big)$ are all weak perfect Bayesian equilibria in Example 5. Let us show that they are also sequential equilibria. To do that, we have to show that the equilibrium belief system is consistent with the equilibrium strategy profile. We distringuish two cases:

Case 1: $\mu^* \in (0, \frac{1}{2}]$. Consider the following sequence of completely mixed strategies for the entrant: $\sigma_{\mathrm{E}}^{(n)} = \frac{\mu^*}{n} P + \frac{(1-\mu^*)}{n} U + \big(1 - \frac{1}{n}\big) A$. Let us check consistency:

1. $\lim_{n \to +\infty} \sigma_{\mathrm{E}}^{(n)} = 0P + 0U + 1A = A$.
2. From Bayes' rule we get:

$$\mu^{(n)} = \frac{\mu^* \frac{1}{n}}{\mu^* \frac{1}{n} + (1 - \mu^*) \frac{1}{n}} = \mu^* \xrightarrow[n \to +\infty]{} \mu^*,$$

hence $\big((A, F); \mu^* \in (0, \frac{1}{2}]\big)$ are all sequential equilibria.

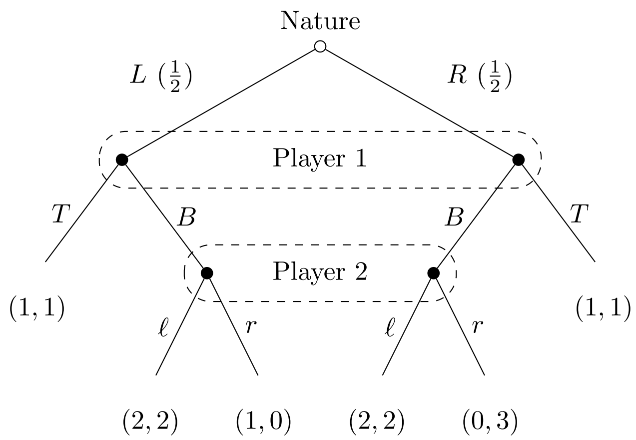
Case 2: $\mu^* = 0$. Consider the following sequence of completely mixed strategies for the entrant: $\sigma_{\mathrm{E}}^{(n)} = \frac{1}{n^2} P + \frac{1}{n} U + \big(1 - \frac{1}{n} - \frac{1}{n^2}\big) A$. Let us check consistency:

1. $\lim_{n \to +\infty} \sigma_{\mathrm{E}}^{(n)} = 0P + 0U + 1A = A$.
2. From Bayes' rule we get:

$$\mu^{(n)} = \frac{\frac{1}{n^2}}{\frac{1}{n^2} + \frac{1}{n}} = \frac{1}{1 + n} \xrightarrow[n \to +\infty]{} 0 = \mu^*,$$

hence $\big((A, F); \mu^* = 0\big)$ is also a sequential equilibrium.

Example 7. *Consider the following extensive-form game:*



Let us find sequential equilibria in Example 7. Consider $\sigma_1^{(n)} = p^{(n)}T + \big(1 - p^{(n)}\big)B$. Bayes' rule then implies:

$$\mu^{(n)} = \frac{\frac{1}{2}\big(1 - p^{(n)}\big)}{\frac{1}{2}\big(1 - p^{(n)}\big) + \frac{1}{2}\big(1 - p^{(n)}\big)} = \frac{1}{2} = \mu^*.$$

Sequential rationality then implies that player 2 will choose ℓ:

$$\ell : 2\frac{1}{2} + 2\frac{1}{2} = 2,$$
$$r : 0\frac{1}{2} + 3\frac{1}{2} = 1.5,$$

and player 1 will therefore choose B. $\big((B, \ell); \mu^* = \frac{1}{2}\big)$ is a sequential equilibrium.

1.4 Important facts about extensive-form equilibrium concepts

1. Every finite extensive-form game has at least one sequential equilibrium.
2. If (σ, μ) is a sequential equilibrium, then (σ, μ) is a weak PBE.
3. If (σ, μ) is a sequential equilibrium, then σ is a subgame-perfect equilibrium.

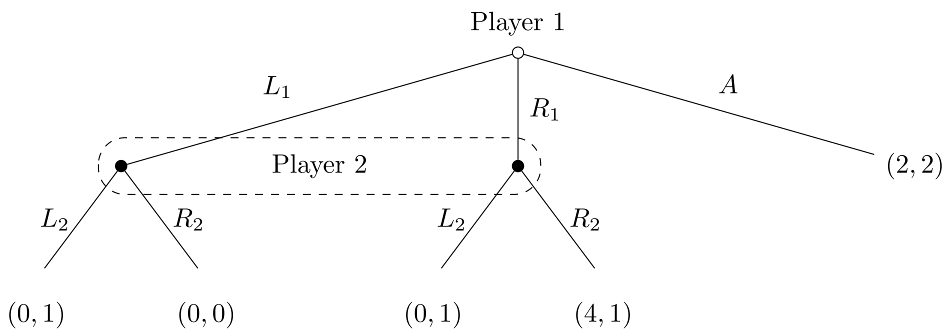
The relationships between the equilibrium concepts can be summarized as follows:

$$\text{Sequential} \subseteq \text{wPBE} \subseteq \text{Nash},$$
$$\text{Sequential} \subseteq \text{SPE} \subseteq \text{Nash}.$$

Remember, however, that wPBE $\not\subseteq$ SPE (Example 3) and SPE $\not\subseteq$ wPBE (Example 2 of Lecture #5).

1.5 Two more examples

Example 8 (Fudenberg and Tirole, 1991; Kohlberg and Mertens, 1986). *Consider the following extensive-form game:*



Observe that in Example 8 there is a sequential equilibrium, in which player 1 plays A. To see that, let μ be the belief that player 2 assigns to history L_1 and consider the expected payoffs of player 2:

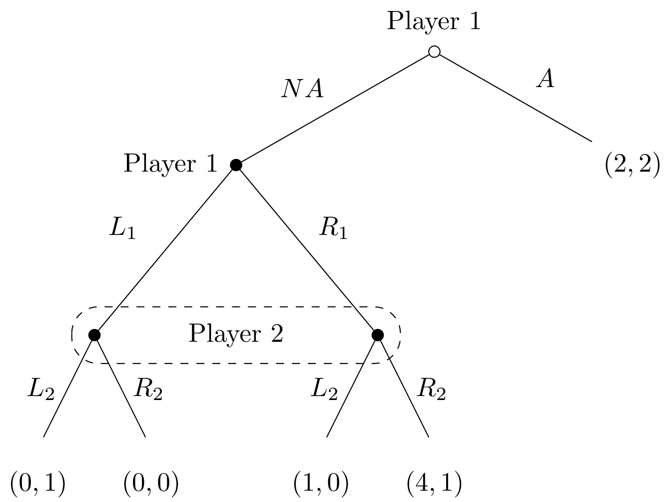
$$L_2 : 1\mu + 0(1 - \mu) = \mu,$$
$$R_2 : 0\mu + 1(1 - \mu) = 1 - \mu.$$

Take e.g. $\mu^* = \frac{3}{4}$, in this case player 2 chooses L_2, and player 1 chooses A. To see that $\mu^* = \frac{3}{4}$ is consistent with (A, L_2) consider the following sequence of completely mixed strategies: $\sigma_1^{(n)} = \frac{3}{4}\epsilon^n L_1 + \frac{1}{4}\epsilon^n R_1 + (1 - \epsilon^n)A$ for $0 < \epsilon < 1$. Observe that

1. $\lim_{n\to+\infty} \sigma_1^{(n)} = 0L_1 + 0R_1 + 1A = A$.
2. $\mu^{(n)} = \frac{\frac{3}{4}\epsilon^n}{\frac{3}{4}\epsilon^n + \frac{1}{4}\epsilon^n} = \frac{3}{4} = \mu^*$.

The next example shows that a small change in the game tree can dramatically change the set of sequentially equilibria:

Example 9 (Fudenberg and Tirole, 1991; Kohlberg and Mertens, 1986). *Consider the following extensive-form game:*



Let us show that playing A cannot be part of a subgame-perfect Nash equilibrium in Example 9. Consider the only proper subgame of Example 9, its strategic form is:

	L_2	R_2
L_1	$0, 1$	$0, 0$
R_1	$1, 0$	$4, 1$

The unique Nash equilibrium of this subgame is (R_1, R_2), hence the unique subgame-perfect equilibrium involves player 1 playing NA, which implies that playing A cannot be part of any sequential equilibrium of this game either.