Game Theory, Spring 2024

Lecture # 6

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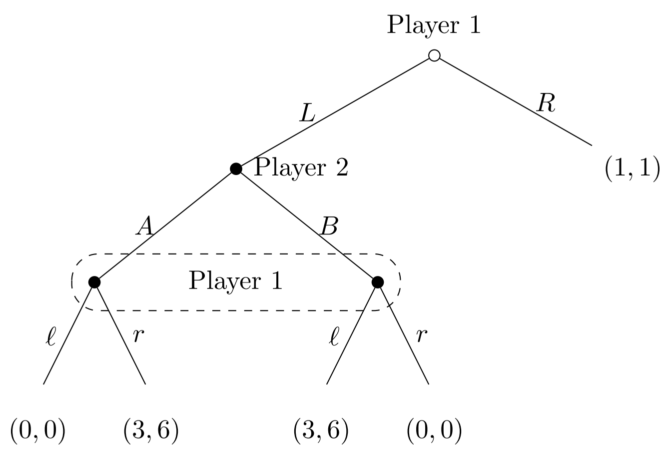
1 Mixed and behavior strategies

Example 4. *Consider the following extensive-form game:*



Recall that a pure strategy is a function $\sigma_i : I_i \mapsto \sigma_i(I_i) \in A(I_i)$ that maps an information set to an action available in this information set. In Example 4, the set of pure strategies for player 1 is $S_1 = \{L\ell, Lr, R\ell, Rr\}$, where $L\ell$ stands for $\sigma_1\big(\{\emptyset\})\big) = L$ and $\sigma_1\big(\{LA, LB\}\big) = \ell$, and Lr stands for $\sigma_1\big(\{\emptyset\})\big) = L$ and $\sigma_1\big(\{LA, LB\}\big) = r$ etc. The definition of a mixed strategy is standard:

Definition 1 (**Mixed strategy**). *A mixed strategy is a probability distribution over pure strategies.*

In Example 4, the following is a mixed strategy: $\frac{1}{4}L\ell + \frac{1}{4}Lr + \frac{1}{4}R\ell + \frac{1}{4}Rr$.

In extensive-form games, it is often more convenient to think about randomization in terms of behavior strategies:

Definition 2 (**Behavior strategy**)**.** *A behavior strategy is a function that maps each information set into a probability distribution over the actions available at that information set, i.e.* $\sigma_i : I_i \mapsto \sigma_i(I_i) \in \Delta\big(A(I_i)\big)$.

In Example 4, the following is a behavior strategy:

$$\sigma_1\big(\{\emptyset\}\big) = \frac{2}{3}L + \frac{1}{3}R \text{ and } \sigma_1\big(\{LA, LB\}\big) = \frac{1}{2}\ell + \frac{1}{2}r.$$

Mixed and behavior strategies are equivalent in games of *perfect recall.*

1.1 Weak perfect Bayesian equilibria in mixed/behavior strategies

Let us find a weak perfect Bayesian equilibrium in mixed strategies in the game of Example 4. Suppose player 1 believes that she is at history LA with probability μ and at history LB with probability $1-\mu$. We will construct an equilibrium, in which player 1 randomizes between ℓ and r according to $p\ell + (1-p)r$. The expected payoffs of player 1 are:

$$\begin{aligned} \ell:&\ \ 0\mu + 3(1-\mu) = 3(1-\mu),\\ r:&\ \ 3\mu + 0(1-\mu) = 3\mu. \end{aligned}$$

By indifference, we have $3(1-\mu^*) = 3\mu^*$, hence $\mu^* = \frac{1}{2}$. Suppose the information set $\{LA, LB\}$ is reached with positive probability. Bayes' rule then implies that player 2 plays $\frac{1}{2}A + \frac{1}{2}B$. Player 2 therefore has to be indifferent between A and B:

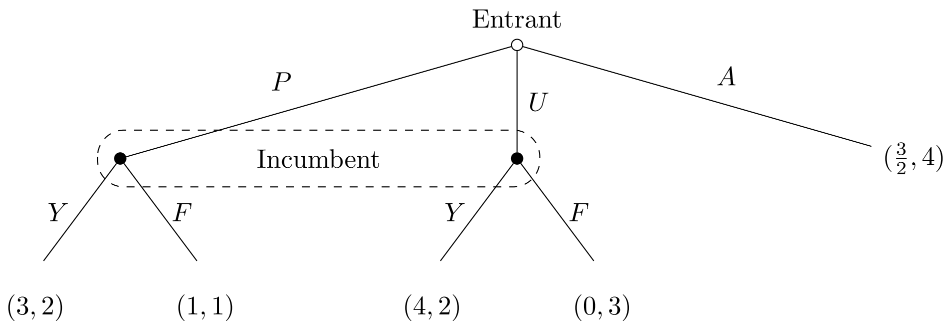
$$\begin{aligned} \ell:&\ \ 0p + 6(1-p) = 6(1-p),\\ r:&\ \ 6p + 0(1-p) = 6p. \end{aligned}$$

By indifference we have $6(1-p^*) = 6p^*$, hence $p^* = \frac{1}{2}$.

If player 1 plays L, here expected payoff is $3\mu^* = 1.5$, which is higher than the payoff from R, hence player 1 plays L and the information set $\{LA, LB\}$ is indeed achived with positive probability, this the following is a weak perfect Bayesian equilibrium:

$$\left(\sigma_1\big(\{\emptyset\}\big) = L, \sigma_1\big(\{LA, LB\}\big) = \frac{1}{2}\ell + \frac{1}{2}r, \sigma_2\big(\{L\}\big) = \frac{1}{2}A + \frac{1}{2}B;\ \mu^* = \frac{1}{2}\right).$$

Example 5. *Consider the following extensive-form game:*



Let us determine the weak perfect Bayesian equilibria of the game in Example 5. Suppose the incumbent believes that she is at history P with probability μ and at history U with probability $1-\mu$. The expected payoffs of the incumbent are then given by:

$$\begin{aligned} Y &:\ 2\mu + 2(1-\mu) = 2, \\ F &:\ 1\mu + 3(1-\mu) = 3 - 2\mu. \end{aligned}$$

It is optimal to choose Y whenever $\mu \geq \frac{1}{2}$, and vice versa. Iff $\mu = \frac{1}{2}$, the incumbent is indifferent between Y and F. We consider three cases.

Case 1: the incumbent plays Y, hence $\mu^* \geq \frac{1}{2}$. In this case, the entrant will play U and the information set $\{P, U\}$ will be reached with probability 1. Bayes' rule then implies $\mu^* = 0$, which is a contradiction, hence there is no such weak perfect Bayesian equilibrium.

Case 2: the incumbent plays F, hence $\mu^* \leq \frac{1}{2}$. In this case, the entrant will play A and the information set set $\{P, U\}$ will be reached with probability 0, hence $\big((A, F), \mu^* \in [0, \frac{1}{2}]\big)$ are weak perfect Bayesian equilibria.

Case 3: the incumbent randomizes according to $pY + (1-p)F$, hence $\mu^* = \frac{1}{2}$.

The expected utilities of the entrant are given by:

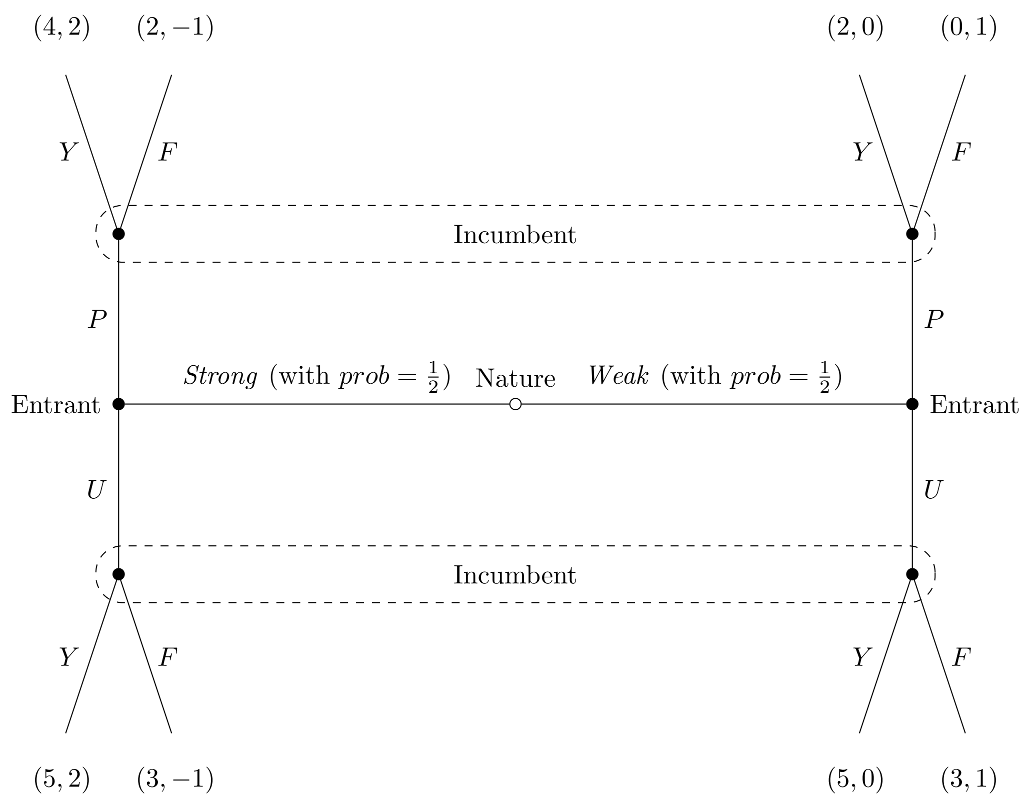
$$P: \ 3p + 1(1-p) = 2p + 1,$$
$$U: \ 4p + 0(1-p) = 4p,$$
$$A: \frac{3}{2}.$$

We distinguish two subcases:

- **Case 3.1:** the information set $\{P, U\}$ is reached with positive probabilty. Bayes' rule then implies that the entrant plays $qP + qU + (1-2q)A$ for some $q > 0$, hence the entrant has to be indifferent between P and U, which is guaranteed whenever $2p^* + 1 = 4p^*$ or $p^* = \frac{1}{2}$ with the resulting payoff of 2, which exceeds the payoff from A, implying that $q^* = \frac{1}{2}$. $\left((\frac{1}{2}P + \frac{1}{2}U, \frac{1}{2}Y + \frac{1}{2}F); \mu^* = \frac{1}{2}\right)$ is a weak perfect Bayesian equilibrium.
- **Case 3.2:** the information set $\{P, U\}$ is reached with probability 0. The entrant then plays A. It is optimal for the entrant to play A whenever $\frac{3}{2} \geq 2p^* + 1$ and $\frac{3}{2} \geq 8p^*$, which is equivalent to $p^* \leq \frac{1}{4}$. Hence for every $p^* \in [0, \frac{1}{4}]$ the following is a weak perfect Bayesian equilibrium: $\left((L, p^*Y + (1-p^*)F); \mu^* = \frac{1}{2}\right)$.

2 Signaling games

Example 6. *Consider the following signaling game:*



The formal defintion of the game in Example 6 is as follows:

Definition 3. *The signaling game in Example 6 consists of the following:*

1. *Players:* $\mathcal{N} = \{\text{Entrant}, \text{Incumbent}\}$.
2. *Histories:* $\mathcal{H} = \{\emptyset, S, W, SP, SPY, SPF, SU, SUY, SUF, WP, WPY, WPF, WU, WUY, WUF\}$.

   Terminal histories: $\mathcal{Z} = \{SPY, SPF, SUY, SUF, WPY, WPF, WUY, WUF\}$.
3. *Player function:* $\mathscr{P} : \mathcal{H} \setminus \mathcal{Z} \mapsto \mathcal{N} \cup \{\text{Nature}\}$.

$$\begin{aligned}
&\mathscr{P}(\emptyset) = \text{Nature}, \\
&\mathscr{P}(S) = \mathscr{P}(W) = \text{Entrant}, \\
&\mathscr{P}(SP) = \mathscr{P}(SU) = \mathscr{P}(WP) = \mathscr{P}(WU) = \text{Incumbent}.
\end{aligned}$$

4. *Exogenous uncertainty: for every* h *such that* $\mathscr{P}(h) = \text{Nature}$, *we need to specify* $f(\cdot|h) \in \Delta\big(A(h)\big)$. *Here we have* $f(S|\emptyset) = f(W|\emptyset) = \frac{1}{2}$.

5. *Collections of information sets for each player:* $\mathcal{I}_{\text{Entrant}} = \big\{\{S\},\{W\}\big\}$ *and* $\mathcal{I}_{Incumbent} = \big\{\{SU, WU\},\{SP, WP\}\big\}$.
6. *Payoff functions* $u_i : \mathcal{Z} \to \mathbb{R}$, *which map terminal histories to payoff for each player* $i \in \mathcal{N}$ *(see the game tree for the payoffs).*

2.1 Separating equilibria

In a separating equilibrium, different types take different actions. Observe that the weak type will never play P, hence we are looking for a separating equilibrium, in which the weak type plays U and the strong type plays P. Since both information sets are reached with positive probabilty, the beliefs at both information sets are derived via Bayes' rule: $\mu^*(Strong|P) = \mu^*(Weak|U) = 1$. If the incumbent observes P, then her optimal response is Y. If the incumbent observes U, then her optimal response is F. The entrant has no profitable deviations: the weak type never plays P; if the strong type deviates to U, the incumbent will play F in response, and the game will end up at SUF with the payoff of 3 for the strong type as opposed to the payoff of 4 from playing P. Hence the following is a weak perfect Bayesian equilibrium:

$$\Big(\sigma_{\text{E}}(W) = U, \sigma_{\text{E}}(S) = P, \sigma_{\text{I}}\big(\{SU, WU\}\big) = F, \sigma_{\text{I}}\big(\{SP, WP\}\big) = Y;\ \mu^*(Strong|P) = \mu^*(Weak|U) = 1\Big)$$

2.2 Pooling equilibria

In a pooling equilibrium, all types take the same action. Since the weak type never plays P, we are looking for pooling equilibria, in which both types play U. Since both types play U, the information set $\{SU, WU\}$ is reached with positive probability, and the beliefs at this information set are derived via Bayes' rule: $\mu^*(Strong|U) = \mu^*(Weak|U) = \frac{1}{2}$. The expected payoffs of the incumbent at $\{SU, WU\}$ are

$$\begin{aligned} Y &: \ 2\frac{1}{2} + 0\frac{1}{2} = 1, \\ F &: \ -1\frac{1}{2} + \frac{1}{2}1 = 0. \end{aligned}$$

The incumbent will therefore choose Y. The entrant has no profitable deviations: the weak never plays P, and the strong type gets 5 at SUY, which is the highest possible payoff for the entrant in this game.

It remains to determine the behavior and the beliefs of the incumbent at the information set $\{SP, WP\}$. Let $\mu^* \equiv \mu^*(Strong|P)$, the expected payoffs of the incumbent are:

$$Y:\ 2\mu^* + 0(1-\mu^*) = 2\mu^*,$$
$$F:\ -1\mu^* + \frac{1}{2}(1-\mu^*) = 1 - 2\mu^*.$$

It is optimal for the incumbent to choose Y for $\mu^* \in [\frac{1}{4}, 1]$ and vice versa. Thus we get two kinds of pooling equilibria:

$$\Big(\sigma_{\mathrm{E}}(W) = \sigma_{\mathrm{E}}(S) = U, \sigma_{\mathrm{I}}\big(\{SU, WU\}\big) = Y, \sigma_{\mathrm{I}}\big(\{SP, WP\}\big) = Y;\ \mu^*(Strong|U) = \frac{1}{2}, \mu^*(Strong|P) \in [\frac{1}{4}, 1]\Big),$$
$$\Big(\sigma_{\mathrm{E}}(W) = \sigma_{\mathrm{E}}(S) = U, \sigma_{\mathrm{I}}\big(\{SU, WU\}\big) = Y, \sigma_{\mathrm{I}}\big(\{SP, WP\}\big) = F;\ \mu^*(Strong|U) = \frac{1}{2}, \mu^*(Strong|P) \in [0, \frac{1}{4}]\Big).$$

2.3 Semi-separating equilibria

We will construct a semi-separating equilibrium, in which the weak type plays U (note that the weak type will never play P, and hence cannot mix) and the strong plays $pP + (1-p)U$ for some $0 < p < 1$. The beliefs of the incumbent are as follows:

$$\begin{aligned}
\mu^*(Strong|P) &= 1 \\
\mu^*(Strong|U) &= \frac{prob(U|Strong)prob(Strong)}{prob(U|Strong)prob(Strong) + prob(U|Weak)prob(Weak)} \\
&= \frac{(1-p)\frac{1}{2}}{(1-p)\frac{1}{2} + 1\frac{1}{2}} = \frac{1-p}{2-p}.
\end{aligned}$$

We therefore have $\mu^*(Weak|U) = 1 - \frac{1-p}{2-p} = \frac{1}{2-p}$.

Let's consider the actions of the incumebent. If the incumbent observes P, then the incumbent will believe that the entrant's type is $Strong$, and will choose Y. Suppose that then incumbent plays $qY + (1-q)F$ after observing U. The entrant mixes between P and U, and therefore has to be indifferent between P and U:

$$P: 4,$$
$$U: 5q + 3(1-q) = 2q + 3.$$

From indifference, we get $q^* = \frac{1}{2}$, hence the incumbent has to be indifferent between

Y and F:

$$Y : \frac{1-p}{2-p}2 + \frac{1}{2-p}0 = \frac{2-2p}{2-p},$$
$$F : \frac{1-p}{2-p}(-1) + \frac{1}{2-p}1 = \frac{p}{2-p}.$$

From indifference, we have $p^* = \frac{2}{3}$, hence $\mu^*(Strong|U) = \frac{1-p^*}{2-p^*} = \frac{1-2/3}{2-2/3} = \frac{1}{4}$. We have constructed the following weak perfect Bayesian equilibrium:

$$\Big(\sigma_{\mathrm{E}}(W) = U, \sigma_{\mathrm{E}}(S) = \frac{2}{3}P + \frac{1}{3}U,\ \sigma_{\mathrm{I}}\big(\{SU, WU\}\big) = \frac{1}{2}Y + \frac{1}{2}F, \sigma_{\mathrm{I}}\big(\{SP, WP\}\big) = Y;$$
$$\mu^*(Strong|P) = 1,\ \mu^*(Strong|U) = \frac{1}{4}\Big).$$