Game Theory, Spring 2024

Lecture # 4

Daniil Larionov

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1 Second-price sealed-bid auctions

In a second-price sealed-bid auction, the highest bidder wins and pays the second-highest bid. We can formally define it as follows:

Definition 1 (**Second-price sealed-bid auction**). *A second-price sealed-bid auction is a Bayesian game that consists of the following:*

1. *Players:* {Bidder 1, ..., Bidder I},
2. *Actions:* $A_1 = \cdots = A_I = \mathbb{R}_+$,
3. *Types:* $\Theta_1 = \cdots = \Theta_I = [0, 1]$,
4. *Probability distribution over type profiles:*

$$\mathbb{P}\big[V_1 \leq v_1, \ldots, V_I \leq v_I\big] = F(v_1) \times \cdots \times F(v_I),$$

5. *Payoffs:*

$$u_i(b_i, b_{-i}; v_i) = \begin{cases} v_i - \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j, \\ \frac{1}{\#win}(v_i - \max_{j \neq i} b_j) & \text{if } b_i = \max_{j \neq i} b_j, \\ 0 & \text{otherwise,} \end{cases}$$

where $\#win$ is the number of winners in the auction.

We are going to look at symmetric Bayesian Nash equilibria of this game in pure strategies. A pure strategy is $\beta : [0,1] \to \mathbb{R}_+$, mapping valuations to bids. We are going to show that second-price auctions have equilibria in (weakly) dominant strategies:

Definition 2. *A strategy profile* $(\beta_1, \ldots, \beta_I)$ *is a Bayesian Nash equilibrium in (weakly) dominant strategies, if for every bidder* i *and for every* v_i*,* b_i *and* b_{-i} *we have*

$$u_i\big(\beta_i(v_i), b_{-i}; v_i\big) \geq u_i(b_i, b_{-i}; v_i).$$

We establish the following proposition:

Proposition 1. *A second-price sealed-bid auction has a Bayesian Nash equilibrium in dominant strategies, in which every bidder bids her own valuation, i.e.* $\beta(v_i) = v_i$*.*

Proof. We show first that bidding $\beta(v_i) = v_i$ weakly dominates bidding any $b_i > v_i$. Let $\overline{b}_{-i} \equiv \max_{j \neq i} b_j$ and consider the following cases:

	$\overline{b}_{-i} < v_i < b_i$	$\overline{b}_{-i} = v_i < b_i$	$v_i < \overline{b}_{-i} < b_i$	$v_i < \overline{b}_{-i} = b_i$	$v_i < b_i < \overline{b}_{-i}$
$\beta(v_i) = v_i$	i wins, and gets $v_i - \overline{b}_{-i}$	i is one of the winners, gets 0	i loses and gets 0	i loses and gets 0	i loses and gets 0
$b_i > v_i$	i wins, and gets $v_i - \overline{b}_{-i}$	i wins, gets 0	i wins, gets $v_i - \overline{b}_{-i} < 0$	i is one of the winners, and gets $\frac{1}{\#win}(v_i - \overline{b}_{-i}) < 0$	i loses and gets 0

Showing that $\beta(v_i) = v_i$ weakly dominates bidding any $b_i < v_i$ is left for you as an exercise (see Exercise 1.1 in Problem Set #3). □

Revenue achieved by the seller in this equilibrium is given by $R^* = V^{(2)}$, where

$V^{(2)}$ is the second-highest value in $\{V_1, \ldots, V_I\}$. The cdf of $V^{(2)}$ is given by:

$$
\begin{aligned}
H(x) = \mathbb{P}\left[V^{(2)} \le x\right] = & \mathbb{P}\left[V_1 \le x, V_2 \le x, \ldots, V_{I-1} \le x, V_I \le x\right] \\
& + \mathbb{P}\left[V_1 > x, V_2 \le x, \ldots, V_{I-1} \le x, V_I \le x\right] \\
& + \mathbb{P}\left[V_1 \le x, V_2 > x, \ldots, V_{I-1} \le x, V_I \le x\right] \\
& + \ldots \\
& + \mathbb{P}\left[V_1 \le x, V_2 \le x, \ldots, V_{I-1} > x, V_I \le x\right] \\
& + \mathbb{P}\left[V_1 \le x, V_2 \le x, \ldots, V_{I-1} \le x, V_I > x\right] \\
= & \left[F(x)\right]^I + I\left[F(x)\right]^{I-1}\left[1 - F(x)\right].
\end{aligned}
$$

The density of $V^{(2)}$ is $h(x) = H'(x) = I(I-1)\left[F(x)\right]^{I-2}\left[1 - F(x)\right]f(x)$. The expected revenue is:

$$
\mathbb{E}\, R^* = \int_0^1 xI(I-1)\left[F(x)\right]^{I-2}\left[1 - F(x)\right]f(x)dx.
$$

Example 1. *Suppose V_i is uniformly distributed on $[0, 1]$ for each i, we then have $F(x) = x$ and $f(x) = 1$. The equilibrium expected revenue is:*

$$
\mathbb{E}\, R^* = \int_0^1 xI(I-1)x^{I-2}\left[1 - x\right]1dx = \frac{I-1}{I+1},
$$

i.e. the same as the equilibrium expected revenue achieved by the corresponding first-price auction, which is not just a coincidence but a consequence of the Revenue Equivalence theorem, which we will not formally prove here. The Revenue Equivalence theorem implies that any Bayesian equilibrium in strictly increasing strategies of any standard auction[1] yields the same expected revenue for the seller as long as bidders' values are independent and identically distributed.

[1] An auction is standard if the highest bidder gets the object.

2 All-pay auctions

In an all-pay auction, the highest bidder wins and everybody pays their own bid. We can formally define it as follows:

Definition 3 (**All-pay auction**). *An all-pay auction is a Bayesian game that consists of the following:*

1. *Players:* $\{\text{Bidder } 1, \ldots, \text{Bidder } I\}$,
2. *Actions:* $A_1 = \cdots = A_I = \mathbb{R}_+$,
3. *Types:* $\Theta_1 = \cdots = \Theta_I = [0, 1]$,
4. *Probability distribution over type profiles:*

$$\mathbb{P}\big[V_1 \leq v_1, \ldots, V_I \leq v_I\big] = F(v_1) \times \cdots \times F(v_I),$$

5. *Payoffs:*

$$u_i(b_i, b_{-i}; v_i) = \begin{cases} v_i - b_i & \text{if } b_i > \max_{j \neq i} b_j, \\ \frac{1}{\#win}(v_i - b_i) & \text{if } b_i = \max_{j \neq i} b_j, \\ -b_i & \text{otherwise,} \end{cases}$$

where #win is the number of winners in the auction.

We are going to look at symmetric Bayesian Nash equilibria of this game in pure strategies. A pure strategy is $\beta : [0, 1] \to \mathbb{R}_+$, mapping valuations to bids. Suppose β is strictly increasing, continuosly differentiable, and $\beta(0) = 0$. Suppose bidder i has valuation v_i and bids b_i. The expected utility of bidder i is then given by:

$$\mathbb{P}\big[\text{win with } b_i \text{ against } \beta\big] v_i - b_i.$$

The winning probabilty $\mathbb{P}\big[\text{win with } b_i \text{ against } \beta\big]$ is equal to $\big[F\big(\beta^{-1}(b_i)\big)\big]^{I-1}$. Define $G(x) \equiv \big[F\big(x\big)\big]^{I-1}$, and let $g(x) \equiv G'(x)$. We can then write down the expected utility of bidder i as follows:

$$G\big(\beta^{-1}(b_i)\big) v_i - b_i.$$

Taking the first-order condition with respect to b_i, we get:

$$g\big(\beta^{-1}(b_i)\big)\big[\beta^{-1}\big]'(b_i)v_i - 1 = 0$$

In equilibrium, we must have $b_i = \beta(v_i)$, hence we get:

$$g(v_i)\frac{1}{\beta'(v_i)}v_i - 1 = 0,$$

which implies that

$$\beta(v_i) = \int_0^{v_i} xg(x)dx.$$

Recall that $g(x) = G'(x) = \frac{\partial}{\partial x}\big[F\big(x\big)\big]^{I-1} = (I-1)\big[F\big(x\big)\big]^{I-2}f(x)$, hence we can rewrite $\beta(v_i)$ as follows:

$$\beta(v_i) = \int_0^{v_i} x(I-1)\big[F\big(x\big)\big]^{I-2}f(x)dx.$$

Example 2. *Suppose V_i is uniformly distributed on $[0,1]$ for each i, we then have $F(x) = x$ and $f(x) = 1$. The equilibrium bidding strategy is then given by:*

$$\beta(v_i) = \int_0^{v_i} x(I-1)x^{I-2}1dx = \frac{I-1}{I}v_i^I.$$

Revenue achieved by the seller in equilibrium is given by $R^* = \sum_{i=1}^{I}\beta(V_i)$. Expected revenue can then be written as:

$$\mathbb{E}\,R^* = \mathbb{E}\sum_{i=1}^{I} V_i = I\,\mathbb{E}\,\beta(V_1) = I\int_0^1 \beta(x)f(x)dx.$$

In the uniform case, we get

$$\mathbb{E}\,R^* = I\int_0^1 \frac{I-1}{I}x^I 1dx = \frac{I-1}{I+1}.$$