Game Theory, Spring 2024

Lecture # 2

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1 Cournot duopoly with incomplete information

We consider the following strategic situation: there are two firms competing in quantities. The inverse demand function is given by $P(q_1, q_2) = \max\{\alpha - q_1 - q_2,\ 0\}$, where $\alpha > 0$. Each firm can have low marginal cost c_L (with probability π_L) or high marginal cost c_H (with probability π_H), independently of the other firm. A firm knows its own marginal cost, but does not know the marginal cost of its competitor.

Definition 1 (**Cournot duopoly with incomplete information**). *This strategic situation defines a Bayesian game that consists of the following:*

1. *Players:* $\{\text{Firm } 1\ ,\ \text{Firm } 2\}$,
2. *Actions:* $A_1 = A_2 = \mathbb{R}_+$,
3. *Types:* $\Theta_1 = \Theta_2 = \{c_L, c_H\}$,
4. *Probability distribution over type profiles:* $prob(c_H, c_H) = \pi_H^2$, $prob(c_L, c_H) = prob(c_H, c_L) = \pi_L \pi_H$, $prob(c_L, c_L) = \pi_L^2$,
5. *Payoffs:* $u_i(q_i, q_{-i}; c_i) = \max\{\alpha - q_i - q_{-i},\ 0\} q_i - c_i q_i$.

We are going to look for a symmetric interior Bayesian Nash equilibrium in pure strategies. An equilibrium is symmetric if both firms play the same equilibrium strategy. An equilibrium is interior if all prices and quantities are strictly positive.

Let us use $\sigma^* = (q_L^*, q_H^*)$ to denote the equilibrium strategy. If a low-type firm plays q_L in response to σ^*, then it gets:

$$\begin{aligned} u_i(q_L, \sigma^*; c_L) &= \pi_L\big[(\alpha - q_L - q_L^*)q_L - c_L q_L\big] + \pi_H\big[(\alpha - q_L - q_H^*)q_L - c_L q_L\big] \\ &= (\alpha - c_L - q_L - \pi_L q_L^* - \pi_H q_H^*)q_L. \end{aligned}$$

The low-type firm's first-order condition with respect to q_L is given by:

$$\alpha - c_L - 2q_L - \pi_L q_L^* - \pi_H q_H^* = 0.$$

Since $\frac{\partial^2 u_i(q_L, \sigma^*; c_L)}{\partial q_L^2} = -2 < 0$, the solution to the low-type firm's first-order condition is the global maximizer of the low-type firm's utility function.

Likewise, if a high-type firm plays q_H in response to σ^*, then it gets:

$$\begin{aligned} u_i(q_H, \sigma^*; c_H) &= \pi_L\big[(\alpha - q_H - q_L^*)q_H - c_H q_H\big] + \pi_H\big[(\alpha - q_H - q_H^*)q_H - c_H q_H\big] \\ &= (\alpha - c_H - q_H - \pi_L q_L^* - \pi_H q_H^*)q_H. \end{aligned}$$

The high-type firm's first-order condition with respect to q_H is given by:

$$\alpha - c_H - 2q_H - \pi_L q_L^* - \pi_H q_H^* = 0.$$

Since $\frac{\partial^2 u_i(q_H, \sigma^*; c_H)}{\partial q_H^2} = -2 < 0$, the solution to the high-type firm's first-order condition is the global maximizer of the high-type firm's utility function.

Since both firms play the same strategy in equilibrium, we must have $q_L = q_L^*$ and $q_H = q_H^*$. We then obtain (by combining the two first-order conditions):

$$\begin{cases} \alpha - c_L - 2q_L^* - \pi_L q_L^* - \pi_H q_H^* = 0, \\ \alpha - c_H - 2q_H^* - \pi_L q_L^* - \pi_H q_H^* = 0, \end{cases}$$

which can be rewritten as:

$$\begin{cases} \alpha - c_L = (2 + \pi_L)q_L^* + \pi_H q_H^*, \\ \alpha - c_H = \pi_L q_L^* + (2 + \pi_H)q_H^*. \end{cases}$$

This is a system of linear equations with two equations and two unknowns: q_L^* and q_H^*. Its solution is given by:

$$\begin{cases} q_L^* = \frac{1}{3}(\alpha - c_L) + \frac{\pi_H}{6}(c_H - c_L), \\ q_H^* = \frac{1}{3}(\alpha - c_H) - \frac{\pi_L}{6}(c_H - c_L). \end{cases}$$

We now establish the following claim:

Claim 1. *A low-type firm produces more than a high-type firm, i.e.* $q_L^* > q_H^*$.

Proof. $q_L^* > q_H^*$ can be equivalently written as:

$$\begin{aligned} &\frac{1}{3}(\alpha - c_L) + \frac{\pi_H}{6}(c_H - c_L) > \frac{1}{3}(\alpha - c_H) - \frac{\pi_L}{6}(c_H - c_L) \\ \Leftrightarrow\ &2(\alpha - c_L) + \pi_H(c_H - c_L) > 2(\alpha - c_H) - \pi_L(c_H - c_L) \\ \Leftrightarrow\ &2(c_H - c_L) + \underbrace{(\pi_L + \pi_H)}_{=1}(c_H - c_L) > 0 \\ \Leftrightarrow\ &3(c_H - c_L) > 0 \Leftrightarrow 3 > 0. \end{aligned}$$

□

To make sure that both quantities are strictly positive, we therefore only have to make sure that $q_H^* > 0$, which is true whenever:

$$\begin{aligned} &\frac{1}{3}(\alpha - c_H) - \frac{\pi_L}{6}(c_H - c_L) > 0 \\ \Leftrightarrow\ &\alpha > c_H + \frac{\pi_L}{2}(c_H - c_L). \end{aligned} \tag{1}$$

To make sure that all the prices are strictly positive, we only have to make sure that $P(q_L^*, q_L^*) = \alpha - 2q_L^* > 0$ since it's the lowest possible price. $\alpha - 2q_L^* > 0$ whenever

$$\begin{aligned} &\alpha > \frac{2}{3}(\alpha - c_L) + \frac{2\pi_H}{6}(c_H - c_L) \\ \Leftrightarrow\ &\alpha > \pi_H(c_H - c_L) - 2c_L. \end{aligned} \tag{2}$$

We now show that Inequality 2 is implied by Inequality 1:

Claim 2. $c_H + \frac{\pi_L}{2}(c_H - c_L) > \pi_H(c_H - c_L) - 2c_L$.

Proof. This inequality can be rewritten as:

$$c_H + 2c_L > \left[\pi_H - \frac{\pi_L}{2}\right](c_H - c_L) \tag{3}$$

The right-hand side of Inequality 3 can then be written as follows:

$$\left[\pi_H - \frac{\pi_L}{2}\right](c_H - c_L) = \frac{2\pi_H - (1-\pi_H)}{2}(c_H - c_L) = \frac{3\pi_H - 1}{2}(c_H - c_L) < \frac{3-1}{2}(c_H - c_L) = c_H - c_L.$$

We therefore have:

$$c_H + 2c_L > c_H - c_L > \left[\pi_H - \frac{\pi_L}{2}\right](c_H - c_L).$$

□

$\alpha > c_H + \frac{\pi_L}{2}(c_H - c_L)$ then guarantees existence of an interior equilibrium.

2 Bertrand duopoly with incomplete information

We consider the following strategic situation: there are two firms competing in prices. The demand function for Firm i is given by

$$D_i(p_i, p_{-i}) = \begin{cases} 1 & \text{if } p_i < p_{-i}, \\ \frac{1}{2} & \text{if } p_i = p_{-i}, \\ 0 & \text{otherwise.} \end{cases}$$

Each firm can have low marginal cost c_L (with probability π_L) or high marginal cost c_H (with probability π_H) independently of the other firm. A firm knows its own marginal cost, but does not know the marginal cost of its competitor.

Definition 2 (Bertrand duopoly with incomplete information). *This strategic situation defines a Bayesian game that consists of the following:*

1. *Players:* {Firm 1 , Firm 2},
2. *Actions:* $A_1 = A_2 = \mathbb{R}_+$,
3. *Types:* $\Theta_1 = \Theta_2 = \{c_L, c_H\}$,

4. *Probability distribution over type profiles:* $prob(c_H, c_H) = \pi_H^2$, $prob(c_L, c_H) = prob(c_H, c_L) = \pi_L \pi_H$, $prob(c_L, c_L) = \pi_L^2$,
5. *Payoffs:*

$$u_i(p_i, p_{-i}; c_i) = \begin{cases} p_i - c_i & \text{if } p_i < p_{-i}, \\ \frac{1}{2}(p_i - c_i) & \text{if } p_i = p_{-i}, \\ 0 & \text{otherwise.} \end{cases}$$

We are going to look for a symmetric Bayesian Nash equilibrium of this game. We first establish the following claim:

Claim 3. *Bertrand duopoly with incomplete information has no symmetric Bayesian Nash equilibria in pure strategies.*

Proof. We prove this claim by contradiction. Suppose that there is a symmetric Bayesian Nash equilibrium in pure strategies, in which both firms play (p_L^*, p_H^*). Let us distinguish three cases.

Case 1: $p_L^* > p_H^*$. In this case, we get the following equilibrium payoffs:

$$\text{For a low type firm: } \pi_L \frac{1}{2}(p_L^* - c_L) + \pi_H 0 = \frac{\pi_L}{2}(p_L^* - c_L),$$
$$\text{For a high type firm: } \pi_L(p_H^* - c_H) + \pi_H \frac{1}{2}(p_H^* - c_H) = \left[\pi_L + \frac{\pi_H}{2}\right](p_H^* - c_H).$$

We therefore must have $p_H^* \geq c_H$ because otherwise a high-type firm would get a strictly negative payoff and could profitably deviate to $p' > p_L^*$ to get zero. But then we have $p_L^* > p_H^* \geq c_H > c_L$, i.e. $p_L^* > c_L$. If a low-type firm deviates to $p_L^* - \epsilon > c_L$ for some small ϵ, then it would get

$$\pi_L(p_L^* - \epsilon - c_L) > \frac{\pi_L}{2}(p_L^* - c_L) \text{ if } \epsilon \text{ is sufficiently small,}$$

hence this low-type firm has a profitable deviation.

Case 2: $p_L^* = p_H^* \equiv p^*$. In this case, we get the following equilibrium payoffs:

$$\text{For a low type firm: } \frac{1}{2}(p^* - c_L),$$
$$\text{For a high type firm: } \frac{1}{2}(p^* - c_H).$$

We therefore must have $p^* \geq c_H$ because otherwise a high-type firm would get a strictly negative payoff and could profitably deviate to $p' > p^*$ to get zero. But then we have $p^* \geq c_H > c_L$, i.e. $p^* > c_L$. If a low-type firm deviates to $p^* - \epsilon > c_L$ for some small ϵ, then it would get

$$(p_L^* - \epsilon - c_L) > \frac{1}{2}(p_L^* - c_L) \ \text{ if } \epsilon \text{ is sufficiently small,}$$

hence this low-type firm has a profitable deviation.

Case 3: $p_H^* > p_L^*$. In this case, we get the following equilibrium payoffs:

$$\begin{aligned} \text{For a low type firm: } & \pi_L \frac{1}{2}(p_L^* - c_L) + \pi_H(p_L^* - c_L) = \left[\frac{\pi_L}{2} + \pi_H\right](p_L^* - c_L), \\ \text{For a high type firm: } & \pi_L 0 + \pi_H \frac{1}{2}(p_H^* - c_H) = \frac{\pi_H}{2}(p_H^* - c_H). \end{aligned}$$

Clearly we must have $p_H^* \geq c_H$. If not, then a high-type firm would get a strictly negative payoff and could profitably deviate to $p' > p_H^*$ to get zero. Likewise, we must have $p_L^* \geq c_L$. If not, then a low-type firm would get a strictly negative payoff and could profitably deviate to $p' > p_H^*$ to get zero. Moreover, we must have $p_L^* = c_L$. Suppose not, for a contradition, i.e. suppose that $p_L^* > c_L$. If a low-type firm deviates to $p^* - \epsilon > c_L$ for some small ϵ, then it would get

$$(p_L^* - \epsilon - c_L) > \left[\frac{\pi_L}{2} + \pi_H\right](p_L^* - c_L) \ \text{ if } \epsilon \text{ is sufficiently small,}$$

hence this low-type firm has a profitable deviation. $p_L^* = c_L$ means that a low-type firm gets zero in any pure-strategy Bayesian-Nash equilibrium of this game (if one exists). But if a low-type firm deviates to p_H^*, then it gets:

$$\frac{\pi_H}{2}(p_H^* - c_L) \geq \frac{\pi_H}{2}(c_H - c_L) > 0,$$

which means that no such equilibrium could exist. □

We will now construct a Bayesian Nash equilibrium in mixed strategies, in which a high-type firm sets $p_H^* = c_H$, and a low-type firms plays a mixed strategy on $[\underline{p}, c_H)$, where $\underline{p}$ is to be determined later. Let $G(p)$ denote the probability that a firm sets a price weakly below p. If a low-type firms sets $p \in [\underline{p}, c_H)$, then it gets:

$$\big(1 - G(p)\big)(p - c).$$

Since the low-type firm plays a mixed strategy, it must be indifferent between all the prices in $[\underline{p}, c_H)$, and moreover for every $p \in [\underline{p}, c_H)$ we have

$$\big(1 - G(p)\big)(p - c_L) = \pi_H(c_H - c_L),$$

which we can now solve for $G(p)$:

$$G(p) = 1 - \pi_H \frac{c_H - c_L}{p - c_L} \text{ for } p \in [\underline{p}, c_H).$$

The complete definition of $G(p)$ is then given by:

$$G(p) = \begin{cases} 0 & \text{for } p < \underline{p}, \\ 1 - \pi_H \frac{c_H - c_L}{p - c_L} & \text{for } p \in [\underline{p}, c_H), \\ 1 & \text{for } p \geq c_H. \end{cases}$$

To determine $\underline{p}$ we solve $G(\underline{p}) = 0$:

$$\begin{aligned} & 0 = 1 - \pi_H \frac{c_H - c_L}{\underline{p} - c_L} \\ \Leftrightarrow\ & \underline{p} - c_L = \pi_H(c_H - c_L) \\ \Leftrightarrow\ & \underline{p} = c_L + \pi_H c_H - \pi_H c_L = (1 - \pi_H)c_L + \pi_H c_H = \pi_L c_L + \pi_H c_H. \end{aligned}$$

Let $F(p)$ be the mixed strategy of a low-type firm. It is then given by:

$$F(p) = \begin{cases} 0 & \text{for } p < \pi_L c_L + \pi_H c_H, \\ \frac{1}{\pi_L}\big[1 - \pi_H \frac{c_H - c_L}{p - c_L}\big] & \text{for } p \in [\pi_L c_L + \pi_H c_H, c_H), \\ 1 & \text{for } p \geq c_H. \end{cases}$$

Claim 4. *Bertrand duopoly with incomplete information has a symmetric Bayesian Nash equilibrium, in which a high-type firm sets $p_H^* = c_H$ and a low-type firm ran-*

domizes on $[\pi_L c_L + \pi_H c_H, c_H)$ *according to* F.

Proof. A high-type firm gets zero in equilibrium, and would not find it profitable to deviate upwards, because it would lead to the payoff of zero as well. It would not find it profitable to deviate downwards either because it could only lead to a negative payoff. A low-type firm is indifferent between all prices in $[\pi_L c_L + \pi_H c_H, c_H)$ by construction, and gets $\pi_H(c_H - c_L)$ in equilibrium. If it deviated to c_H, it would get $\frac{\pi_H}{2}(c_H - c_L) < \pi_H(c_H - c_L)$. If it deviated to $p' > c_H$, then it would get zero, which could not be profitable. If it deviated to $p' < \underline{p} = \pi_L c_L + \pi_H c_H$, then it would get $p' - c_L < \underline{p} - c_L = \pi_H(c_H - c_L)$. $\square$