Game Theory, Spring 2024

Lecture # 12

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1 Extreme points of the set of correlated of equilibria

We have shown that for any strategic-form game $\Gamma = \big(\mathcal{I}, A, \{u_i\}_{i\in\mathcal{I}}\big)$ the set of its correlated equilibria $CE(\Gamma)$ is convex and compact. Such sets can be characterized via their *extreme points.*

Definition 1 (**Extreme point**)**.** *A point α in a convex set C is an extreme point of C if there are no two distinct points $\alpha' \in C$ and $\alpha'' \in C$, and no $\lambda \in (0,1)$ such that $\alpha = \lambda\alpha' + (1-\lambda)\alpha''$. We write $\alpha \in \mathrm{extreme}(C)$.*

Suppose $\alpha \in CE(\Gamma)$, to check whether $\alpha \in \mathrm{extreme}\big(CE(\Gamma)\big)$, do the following:

1. Identify the binding incentive constraints (i.e. satisfied by α as equalities), and let IC^* denote the set of those incentive constraints.
2. Identify the binding non-negativity constraints (i.e. satisfied by α as equalities), and let NN^* denote the set of those non-negativity constraints.
3. Write down the following system of binding constraints:

$$
\begin{aligned}
&\big(\mathrm{IC}^*_{(a_i,\tilde{a}_i)}\big) && \sum_{a_{-i}\in A_{-i}} \alpha(a_i,a_{-i})\big[u_i(a_i,a_{-i}) - u_i(\tilde{a}_i,a_{-i})\big] = 0 \qquad \forall (a_i,\tilde{a}_i) \text{ s.t. } \mathrm{IC}_{(a_i,\tilde{a}_i)} \in IC^*,\\
&\big(\mathrm{NN}^*_a\big) && \alpha(a) = 0 \qquad \forall a \text{ s.t. } \mathrm{NN}_a \in NN^*, \qquad (1)\\
&\big(\mathrm{Prob}\big) && \sum_{a\in A} \alpha(a) = 1.
\end{aligned}
$$

4. Check whether α is the unique solution to the system of binding constraints in (1). We will show below that $\alpha \in \text{extreme}\big(CE(\Gamma)\big)$ if and only if α is the unique solution to System (1).

Example 1. *Consider the strategic-form game Γ_1 from Lecture #11:*

	L	R
T	$4,4$	$1,5$
B	$5,1$	$0,0$

Take $\alpha = (x,y,z,w) = \big(\frac{1}{3},\frac{1}{3},\frac{1}{3},0\big) \in CE(\Gamma_1)$. *Let's check whether* $\alpha \in \text{extreme}\big(CE(\Gamma_1)\big)$.

1. Identify the binding incentive constraints:

$$\begin{aligned}
&(\text{IC}_T) && -x+y = -\frac{1}{3}+\frac{1}{3} = 0,\\
&(\text{IC}_B) && z-w = \frac{1}{3}-0 = \frac{1}{3} > 0,\\
&(\text{IC}_L) && -x+z = -\frac{1}{3}+\frac{1}{3} = 0,\\
&(\text{IC}_R) && y-w = \frac{1}{3}-0 = \frac{1}{3} > 0,
\end{aligned}$$

2. Identify the binding non-negativity constraints:

$$(\text{NN}) \qquad x>0,\ y>0,\ z>0,\ w=0.$$

3. Write down the system of binding constraints:

$$\begin{aligned}
&(\text{IC}^*_T) && -x+y=0,\\
&(\text{IC}^*_L) && -x+z=0,\\
&(\text{NN}^*_{(B,R)}) && w=0,\\
&\big(\text{Prob}\big) && x+y+z+w=1.
\end{aligned}$$

4. Check whether α is the unique solution to the system of binding constraints.

The unique solution to the system of binding constraints is $\alpha = (x, y, z, w) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$*, hence* $\alpha \in \text{extreme}\big(CE(\Gamma_1)\big)$.

Example 2. *Consider the strategic-form game* Γ_1 *from Lecture #11:*

	L	R
T	$4,4$	$1,5$
B	$5,1$	$0,0$

Take $\alpha = (x, y, z, w) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0\right) \in CE(\Gamma_1)$*. Let's check whether* $\alpha \in \text{extreme}\big(CE(\Gamma_1)\big)$.

1. *Identify the binding incentive constraints:*

$$\begin{aligned}
&(\text{IC}_T) \qquad -x + y = -\frac{1}{4} + \frac{1}{4} = 0,\\
&(\text{IC}_B) \qquad z - w = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} > 0,\\
&(\text{IC}_L) \qquad -x + z = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4} > 0,\\
&(\text{IC}_R) \qquad y - w = \frac{1}{4} - 0 = \frac{1}{4} > 0,
\end{aligned}$$

2. *Identify the binding non-negativity constraints:*

$$(\text{NN}) \qquad x > 0,\ y > 0,\ z > 0,\ w = 0.$$

3. *Write down the system of binding constraints:*

$$\begin{aligned}
&(\text{IC}_T^*) \qquad -x + y = 0,\\
&(\text{NN}_{(B,R)}^*) \qquad w = 0,\\
&\big(\text{Prob}\big) \qquad x + y + z + w = 1.
\end{aligned}$$

4. *Check whether* α *is the unique solution to the system of binding constraints.*

   This system has infinitely many solutions, hence $\alpha \notin \text{extreme}\big(CE(\Gamma_1)\big)$.

We establish the following proposition:

Proposition 1. *Let Γ be a strategic-form game, $\alpha \in CE(\Gamma)$, and let $A\alpha = b$ be the matrix form of the system of constraints binding at α. The following are equivalent:*

(I) $\alpha \in \text{extreme}\big(CE(\Gamma)\big)$,

(II) *α is the unique solution to the system of constraints binding at α,*

(III) $\text{rank}A = \#\text{action profiles in } \Gamma \equiv k$.

Proof. We present the proof in a series of lemmas:

Lemma 1. *(II) $\Rightarrow$ (III).*

Proof. Suppose that $\text{rank}A < k$, then the columns of A are linearly dependent, i.e. there exists a non-trivial linear combination

$$\lambda_1 A_1 + \cdots + \lambda_k A_k = 0.$$

Let $\lambda \equiv \big(\lambda_1, \ldots, \lambda_k\big)^T$, and $\alpha' \equiv \alpha + \lambda$, then $A\alpha' = A(\alpha + \lambda) = A\alpha + A\lambda = b + 0 = b$, i.e. α' is also a solution to the system of constraints binding at α. □

Lemma 2. *(III) $\Rightarrow$ (II).*

Proof. Suppose that there are at least two distinct solutions to the system of binding constraints, i.e. there is α' such that $A\alpha = A\alpha' = b$ with $\alpha \neq \alpha'$. Define $\lambda \equiv \alpha - \alpha' \neq 0$, we then have $A(\alpha - \alpha') = A\alpha - A\alpha' = 0$ and, hence the columns of A are linearly dependent and therefore $\text{rank}A < k$. □

Lemma 3. *(II) $\Rightarrow$ (I).*

Proof. Suppose $\alpha \notin \text{extreme}\big(CE(\Gamma)\big)$, then there exist two distinct correlated equilibria, $\alpha' \in CE(\Gamma)$ and $\alpha'' \in CE(\Gamma)$, and some $\lambda \in (0, 1)$ such that $\alpha = \lambda\alpha' + (1 - \lambda)\alpha''$. Let $IC^*(\alpha)$ and $NN^*(\alpha)$ be the incentive and non-negativity constraints binding at α. Since α' and α'' satisfy all the incentive and non-negativity constraints, we have[1]

[1] The same is true for α''.

$$(\text{IC}^*_{(a_i,\tilde{a}_i)}) \qquad \sum_{a_{-i}\in A_{-i}} \alpha'(a_i,a_{-i})\big[u_i(a_i,a_{-i}) - u_i(\tilde{a}_i,a_{-i})\big] = 0 \qquad \forall (a_i,\tilde{a}_i) \text{ s.t. } \text{IC}_{(a_i,\tilde{a}_{-i})} \in IC^*(\alpha),$$

$$(\text{NN}^*_a) \qquad \alpha'(a) = 0 \qquad \forall a \text{ s.t. } \text{NN}_a \in NN^*(\alpha),$$

$$(\text{Prob}) \qquad \sum_{a\in A} \alpha'(a) = 1,$$

thus the system of constraints binding at α has more than one solution. □

Lemma 4. *(I) ⇒ (III).*

Proof. Suppose rank$A < k$, then A has linearly dependent columns, i.e. there exists a non-trivial linear combination

$$\lambda_1 A_1 + \cdots + \lambda_k A_k = 0.$$

Let $\lambda \equiv \big(\lambda_1, \ldots, \lambda_k\big)^T$, and $\alpha' \equiv \alpha + \epsilon\lambda$ and $\alpha'' \equiv \alpha - \epsilon\lambda$ for some small $\epsilon > 0$. Clearly, $A\alpha' = A(\alpha+\epsilon\lambda) = A\alpha + \epsilon A\lambda = b + 0 = b$ and $A\alpha'' = A(\alpha-\epsilon\lambda) = A\alpha - \epsilon A\lambda = b - 0 = b$, thus those incentive and non-negativity constraints that are binding at α are also binding at α' and α''. By continuity, those incentive and non-negativity constraints that are slack at α continue to be slack at α' and α'' (as long as ϵ is small enough), hence $\alpha' \in CE(\Gamma)$ and $\alpha'' \in CE(\Gamma)$, but we then have

$$\alpha = \frac{1}{2}\big(\alpha+\epsilon\lambda\big) + \frac{1}{2}\big(\alpha-\epsilon\lambda\big) = \frac{1}{2}\alpha' + \frac{1}{2}\alpha'',$$

hence $\alpha \notin \text{extreme}\big(CE(\Gamma)\big)$. □

□