Game Theory, Spring 2024

Lecture # 10*

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1 Automaton representation of strategy profiles

Definition 1. *An automaton is a tuple* (W, w^0, f, τ)*, where*

- W *is the set of automaton states,* w^0 *is the initial state of the automaton,*
- $f : W \to A$ *is a decision function,*
- $\tau : W \times A \to W$ *is a transition function.*

We can use automata to represent strategy profiles in infinitely repeated games as in the following examples:

Example 1 (**Grim-trigger, Grim-trigger**).

- *Set of states:* $W = \{w_{cc}, w_{dd}\}$*;* $w^0 = w_{cc}$*;*
- *Decision function:* $f(w_{cc}) = (c, c)$*,* $f(w_{dd}) = (d, d)$*;*
- *Transition function:*

$$\tau(w, a) = \begin{cases} w_{cc} & \text{if } w = w_{cc} \text{ and } a = (c, c); \\ w_{dd} & \text{otherwise.} \end{cases}$$

*These notes are adapted from *"Repeated Games and Reputations"* by George J. Mailath and Larry Samuelson.

Example 2 (*k*-**punishment,** *k*-**punishment**).

- *Set of states:* $W = \{w_{cc}, w_{dd_1}, \dots, w_{dd_k}\}$; $w^0 = w_{cc}$;
- *Decision function:* $f(w_{cc}) = (c, c),\ f(w_{dd_1}) = \cdots = f(w_{dd_k}) = (d, d)$;
- *Transition function:*

$$\tau(w, a) = \begin{cases} w_{cc} & \text{if } \big(w = w_{cc} \text{ and } a = (c, c)\big) \text{ or } w = w_{dd_k}; \\ w_{dd_1} & \text{if } w = w_{cc} \text{ and } a \neq (c, c); \\ w_{dd_2} & \text{if } w = w_{dd_1}; \\ \vdots & \\ w_{dd_k} & \text{if } w = w_{dd_{k-1}}. \end{cases}$$

Suppose the automaton $\big(W, w^0, f, \tau\big)$ represents a strategy profile σ, and use $V_i(w)$ to denote player i's discounted payoff from the play according to $\big(W, w^0, f, \tau\big)$ that begins in state $w \in W$. We can write $V_i(w)$ as follows:

$$V_i(w) = (1 - \delta)u_i\big(f(w)\big) + \delta V_i\Big(\tau\big(w, f(w)\big)\Big).$$

We can establish the following proposition:

Proposition 1. *The strategy profile σ is a subgame-perfect Nash equilibrium if and only if for any $w \in W$ accessible*[1] *from w^0, the action profile $f(w)$ is a Nash equilibrium of the strategic-form game $\mathcal{G}^w \equiv \big(\mathcal{I}, A, \{g_i^w\}_{i \in \mathcal{I}}\big)$, where*

$$g_i^w(a_i, a_{-i}) \equiv (1 - \delta)u_i(a_i, a_{-i}) + \delta V_i\Big(\tau\big(w, (a_i, a_{-i})\big)\Big).$$

Proof. **"If":** Suppose $f(w)$ is a Nash equilibrium of $\mathcal{G}^w$ for all $w \in W$ accessible from w^0. Let $\hat{\sigma}_i$ be a one-shot deviation from σ_i for player i such that $\hat{a}_i = \hat{\sigma}_i(\hat{h}^t) \neq \sigma_i(\hat{h}^t)$

[1] w is accessible from w^0 if there exists a history of play such that, beginning in w^0, the automaton reaches w after that history.

for some history $\hat{h}^t$. The deviating continuation payoff from $\hat{h}^t$ is given by:

$$\begin{aligned}
U_i\big(\hat{\sigma}_i|_{\hat{h}^t}, \sigma_{-i}|_{\hat{h}^t}\big) &= (1-\delta)u_i\big(\hat{a}_i, \sigma_{-i}|_{\hat{h}^t}(\emptyset)\big) + \delta V_i\Big(\tau\big(w, (\hat{a}_i, \sigma_{-i}|_{\hat{h}^t}(\emptyset))\big)\Big) \\
&= (1-\delta)u_i\big(\hat{a}_i, f_{-i}(w)\big) + \delta V_i\Big(\tau\big(w, (\hat{a}_i, f_{-i}(w))\big)\Big) \\
&\leq (1-\delta)u_i\big(f_i(w), f_{-i}(w)\big) + \delta V_i\Big(\tau\big(w, (f_i(w), f_{-i}(w))\big)\Big) \quad [f(w) \text{ is a NE of } \mathcal{G}^w] \\
&= (1-\delta)u_i\big(\sigma_i|_{\hat{h}^t}(\emptyset), \sigma_{-i}|_{\hat{h}^t}(\emptyset)\big) + \delta V_i\Big(\tau\big(w, (\sigma_i|_{\hat{h}^t}(\emptyset), \sigma_{-i}|_{\hat{h}^t}(\emptyset))\big)\Big) \\
&= U_i\big(\sigma_i|_{\hat{h}^t}, \sigma_{-i}|_{\hat{h}^t}\big),
\end{aligned}$$

hence this one-shot deviation is not profitable. The one-shot deviation principle then implies that σ is a subgame-perfect Nash equilibrium.

"Only if": Suppose $f(w)$ is not a Nash equilibrium of $\mathcal{G}^w$ for some $w \in W$ accessible from w^0, then there exists a deviation $\hat{a}_i$ such that

$$(1-\delta)u_i\big(\hat{a}_i, f_{-i}(w)\big) + \delta V_i\Big(\tau\big(w, (\hat{a}_i, f_{-i}(w))\big)\Big) > (1-\delta)u_i\big(f(w)\big) + \delta V_i\Big(\tau\big(w, f(w)\big)\Big). \quad (1)$$

Since w is accessible from w^0, there is a history $\hat{h}^t$ such that the automaton $\big(W, w^0, f, \tau\big)$ ends up in state w after history $\hat{h}^t$. Consider the following one-shot deviation from σ_i:

$$\hat{\sigma}_i(h^\tau) = \begin{cases} \hat{a}_i & \text{if } h^\tau = \hat{h}^t, \\ \sigma_i(h^\tau) \text{ if } h^\tau \neq \hat{h}^t. \end{cases}$$

The deviating continuation payoff from $\hat{h}^t$ is given by:

$$\begin{aligned}
U_i\big(\hat{\sigma}_i|_{\hat{h}^t}, \sigma_{-i}|_{\hat{h}^t}\big) &= (1-\delta)u_i\big(\hat{a}_i, \sigma_{-i}|_{\hat{h}^t}(\emptyset)\big) + \delta V_i\Big(\tau\big(w, (\hat{a}_i, \sigma_{-i}|_{\hat{h}^t}(\emptyset))\big)\Big) \\
&= (1-\delta)u_i\big(\hat{a}_i, f_{-i}(w)\big) + \delta V_i\Big(\tau\big(w, (\hat{a}_i, f_{-i}(w))\big)\Big) \\
&> (1-\delta)u_i\big(f_i(w), f_{-i}(w)\big) + \delta V_i\Big(\tau\big(w, (f_i(w), f_{-i}(w))\big)\Big) \quad [\text{by Inequality (1)}] \\
&= (1-\delta)u_i\big(\sigma_i|_{\hat{h}^t}(\emptyset), \sigma_{-i}|_{\hat{h}^t}(\emptyset)\big) + \delta V_i\Big(\tau\big(w, (\sigma_i|_{\hat{h}^t}(\emptyset), \sigma_{-i}|_{\hat{h}^t}(\emptyset))\big)\Big) \\
&= U_i\big(\sigma_i|_{\hat{h}^t}, \sigma_{-i}|_{\hat{h}^t}\big),
\end{aligned}$$

thus $\hat{\sigma}_i$ is a profitable deviation from σ_i, and σ therefore cannot be a subgame-perfect Nash equilibrium. □

2 Self-generation

Let $\mathcal{E}$ be an arbitrary subset of $\mathcal{F} \equiv conv\big(\{v|v = u(a), a \in A\}\big)$[2], the set of feasible payoff profiles. We introduce the following definitions:

Definition 2 (**Enforceability**). *A pure action profile a^* is enforceable on $\mathcal{E} \subseteq \mathcal{F}$ if there exist continuation payoffs $\gamma : A \to \mathcal{E}$ such that for every player i and every action $a_i \in A_i$ we have:*

$$(1-\delta)u_i(a_i^*, a_{-i}^*) + \delta\gamma_i(a_i^*, a_{-i}^*) \geq (1-\delta)u_i(a_i, a_{-i}^*) + \delta\gamma_i(a_i, a_{-i}^*).$$

Definition 3 (**Decomposability**). *A payoff profile $v \in \mathcal{F}$ is decomposable on $\mathcal{E} \subseteq \mathcal{F}$ if there exists an action profile a^*, enforceable on $\mathcal{E}$, such that for each player i*

$$v_i = (1-\delta)u_i(a_i^*, a_{-i}^*) + \delta\gamma_i(a_i^*, a_{-i}^*),$$

where γ_i is the continuation payoff enforcing a^* on $\mathcal{E}$ for player i.

Definition 4 (**Self-generation**). *$\mathcal{E}$ is self-generating if any payoff profile $v \in \mathcal{E}$ is decomposable on $\mathcal{E}$.*

We establish the following proposition:

Proposition 2. *If $\mathcal{E}$ is self-generating, then $\mathcal{E}$ is a set of subgame-perfect Nash equilibrium payoffs.*

Proof. Since $\mathcal{E}$ is self-generating, for every $v \in \mathcal{E}$ there is a decomposing action profile $\tilde{a}(v)$ and its enforcing continuation payoff $\gamma^v : A \to \mathcal{E}$ such that for every player i and every action $a_i \in A_i$ we have:

$$(1-\delta)u_i\big(\tilde{a}_i(v), \tilde{a}_{-i}(v)\big) + \delta\gamma_i^v\big(\tilde{a}_i(v), \tilde{a}_{-i}(v)\big) \geq (1-\delta)u_i\big(a_i, \tilde{a}_{-i}(v)\big) + \delta\gamma_i^v\big(a_i, \tilde{a}_{-i}(v)\big).$$

Take any arbitrary payoff profile $v^0 \in \mathcal{E}$ and define the following automaton:

- Set of states is $\mathcal{E}$, the initial state is v^0,

[2] $conv\big(\{v|v = u(a), a \in A\}\big)$ denotes the *convex hull* of $\{v|v = u(a), a \in A\}$, the smallest convex set that contains $\{v|v = u(a), a \in A\}$. In other words, $\mathcal{F}$ is the set all of payoff profiles that can be obtained via (possibly) correlated randomizations over all pure action profiles.

- Decision function: $f(v) = \tilde{a}(v)$,
- Transition function: $\tau(v, a) = \gamma^v(a)$.

Let $\left\{\left(v^t, \tilde{a}(v^t)\right)\right\}_{t=0}^{\infty}$ be the sequence of payoff and action profiles that realize when the players play according to the automaton $\left(\mathcal{E}, v^0, \tilde{a}(v), \gamma^v(a)\right)$, i.e. $v^t \equiv \gamma^{v^{t-1}}(a^{t-1})$ for all $t > 0$ and $(v^0, \tilde{a}(v^0))$ is the initial element of the sequence. We then have

$$v_i^0 = (1-\delta)\sum_{\tau=0}^{t-1}\delta^\tau u_i\big(\tilde{a}(v^\tau)\big) + \delta^t v^t \xrightarrow[t\to\infty]{} (1-\delta)\sum_{\tau=0}^{\infty}\delta^\tau u_i\big(\tilde{a}(v^\tau)\big) = V_i(v^0),$$

i.e. v_i^0 is the payoff of player i if the players play according to the automaton $\left(\mathcal{E}, v^0, \tilde{a}(v), \gamma^v(a)\right)$ with the initial state v^0. Since v^0 was arbitrarily chosen, we have for every state $v \in \mathcal{E}$, and every player i and every $a_i \in A_i$

$$v_i = (1-\delta)u_i\big(\tilde{a}_i(v), \tilde{a}_{-i}(v)\big) + \delta V_i\big(\gamma^v\big(\tilde{a}_i(v), \tilde{a}_{-i}(v)\big)\big) \geq (1-\delta)u_i\big(a_i, \tilde{a}_{-i}(v)\big) + \delta V_i\big(\gamma^v\big(a_i, \tilde{a}_{-i}(v)\big)\big).$$

Proposition 1 then implies that for any $v \in \mathcal{E}$ the automaton $\left(\mathcal{E}, v, \tilde{a}(v), \gamma^v(a)\right)$ represents a subgame-perfect Nash equilibrium with the payoff profile v. □

Let us use $\mathcal{E}^*$ to denote the set of all subgame-perfect Nash equilibrium payoffs. The following corollary is immediate:

Corollary 1. *$\mathcal{E}^*$ is the largest self-generating set.*

Example 3. *Consider the following infinitely repeated prisoner's dilemma:*

	c	d
c	$5, 5$	$1, 6$
d	$6, 1$	$2, 2$

We establish the following claim:

Claim 1. *The set $\big\{(2, 2); (5, 5)\big\}$ is self-generating for sufficiently high δ's.*

Proof. Consider $(2,2)$ first. It is enforced by (d,d) as long as the following decomposability and incentive compatibility constraints are satisfied:

$$2 = (1-\delta)2 + \delta\gamma_1(d,d) \geq (1-\delta)1 + \delta\gamma_1(c,d)$$
$$2 = (1-\delta)2 + \delta\gamma_2(d,d) \geq (1-\delta)1 + \delta\gamma_2(d,c)$$

Selecting $\big(\gamma_1(d,d), \gamma_2(d,d)\big) = (2,2)$; $\big(\gamma_1(c,d), \gamma_2(c,d)\big) = (2,2)$, and $\big(\gamma_1(d,c), \gamma_2(d,c)\big) = (2,2)$ makes sure that this is the case for all $\delta \in (0,1)$.

Consider now $(5,5)$. It is enforced by (c,c) as long as the following decomposability and incentive compatibility constraints are satisfied:

$$5 = (1-\delta)5 + \delta\gamma_1(c,c) \geq (1-\delta)6 + \delta\gamma_1(d,c)$$
$$5 = (1-\delta)5 + \delta\gamma_2(c,c) \geq (1-\delta)6 + \delta\gamma_2(c,d)$$

Selecting $\big(\gamma_1(c,c), \gamma_2(c,c)\big) = (5,5)$; $\big(\gamma_1(c,d), \gamma_2(c,d)\big) = (2,2)$, and $\big(\gamma_1(d,c), \gamma_2(d,c)\big) = (2,2)$ makes sure that this is the case as long as $5 \geq (1-\delta)6 + \delta 2$ or $\delta \geq \frac{1}{4}$. □