Supplementary Appendix to Full Surplus Extraction from Colluding Bidders

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A Proof of Proposition 0

Proof. Let us first consider the choices made by the buyers who face a reserve price r. Depending on the reserve price chosen by the seller, there are three possible kinds of continuation games to consider:

Case I: $r \leq \underline{\theta}$

In this case, the following lemma applies:

Lemma 1. *If $r \leq \underline{\theta}$,*

- *any low-type buyer bids his own valuation in equilibrium: $\underline{b} = \underline{\theta}$,*
- *any high-type buyer randomizes his bids on $\big(\underline{\theta}, (1-q^{n-1})\overline{\theta} + q^{n-1}\underline{\theta}\big]$ according to*

$$G(b) = \frac{q}{1-q}\left[\left(\frac{\overline{\theta}-\underline{\theta}}{\overline{\theta}-b}\right)^{\frac{1}{n-1}} - 1\right].$$

The ex ante equilibrium payoff of the buyers is:

$$v^*_{r\leq\underline{\theta}} = (1-q)q^{n-1}(\overline{\theta}-\underline{\theta}).$$

The seller's expected revenue is:

$$\mathcal{R}^*_{r\leq\underline{\theta}} = (1-q^n)\overline{\theta} + q^n\underline{\theta} - n(1-q)q^{n-1}(\overline{\theta}-\underline{\theta}).$$

Proof. It is clear that both types will be willing to participate. It can be easily shown that there is no Nash equilibrium in pure strategies. It is also immediately clear that a low-type buyer will never place a bid higher than his own valuation because winning with such a high bid will lead to a negative payoff. But low types will not place a bid that is lower than their valuation even if they have an opportunity to do so. Suppose low-type buyers do place a bid $r \leq \underline{b} < \underline{\theta}$ in equilirium, then one of them could deviate to $\underline{b} + \epsilon$ and guarantee winning the auction for sure if his competitors are low types as well, hence there is a profitable deviation.

Suppose $\Phi(b)$ is the unconditional distribution of equilibrium bids for every buyer. The expected payoff of a buyer with type $\overline{\theta}$ is given by: $\Phi^{n-1}(b)(\overline{\theta}-b)$. Only low types bid $\underline{\theta}$, hence $\Phi(\underline{\theta}) = q$. By indifference we have:

$$\Phi^{n-1}(b)(\overline{\theta}-b) = q^{n-1}(\overline{\theta}-\underline{\theta}),$$

which means that $\Phi(b) = q\big(\frac{\overline{\theta}-\underline{\theta}}{\overline{\theta}-b}\big)^{\frac{1}{n-1}}$ and $v^*_{r\leq\underline{\theta}} = (1-q)q^{n-1}(\overline{\theta}-\underline{\theta})$.

To find the upper bound of the support we solve $q\left(\frac{\overline{\theta}-\underline{\theta}}{\overline{\theta}-\overline{b}}\right)^{\frac{1}{n-1}} = 1$, which leads to $\overline{b} = (1-q^{n-1})\overline{\theta} + q^{n-1}\underline{\theta}$. Since $\Phi(b)$ is the unconditional distribution of equilibrium bids, the actual mixed strategy of high-type buyers is given by:

$$G(b) \equiv \Phi(b|\theta_i = \overline{\theta}) = \frac{q}{1-q}\left[\left(\frac{\overline{\theta}-\underline{\theta}}{\overline{\theta}-b}\right)^{\frac{1}{n-1}} - 1\right].$$

The equilibrium is efficient, hence it leads to the total surplus given by $(1-q^n)\overline{\theta}+q^n\underline{\theta}$. The resulting revenue of the seller is

$$\begin{aligned}\mathcal{R}^*_{r\leq\underline{\theta}} &= (1-q^n)\overline{\theta} + q^n\underline{\theta} - nv^*_{r\leq\underline{\theta}}\\ &= (1-q^n)\overline{\theta} + q^n\underline{\theta} - n(1-q)q^{n-1}(\overline{\theta}-\underline{\theta}).\end{aligned}$$

□

Case II: $\underline{\theta} < r < \overline{\theta}$

In this case, the following lemma applies:

Lemma 2. *If $\underline{\theta} < r < \overline{\theta}$,*

- *any low-type buyer chooses to abstain: $\underline{b} = \emptyset$,*
- *any high-type buyer randomizes his bids on $\left[r, (1-q^{n-1})\overline{\theta} + q^{n-1}r\right]$ according to*

$$G(b) = \frac{q}{1-q}\left[\left(\frac{\overline{\theta}-r}{\overline{\theta}-b}\right)^{\frac{1}{n-1}} - 1\right].$$

The ex ante equilibrium payoff of the buyers is:

$$v^*_{\underline{\theta}<r<\overline{\theta}} = (1-q)q^{n-1}(\overline{\theta}-r).$$

The expected revenue of the seller is:

$$\mathcal{R}^*_{\underline{\theta}<r<\overline{\theta}} = (1-q^n)\overline{\theta} - n(1-q)q^{n-1}(\overline{\theta}-r).$$

Proof. In this case, only the high-type buyers are willing to participate. It can also be shown that there is no equilibrium in pure strategies. Hence we will be looking for an equilibrium in mixed strategies. Suppose that a high type buyer randomizes his bids according to the distribution function $G(b)$. The payoff of a high type buyer who is bidding b is given by:

$$\begin{aligned}&\left(q^{n-1} + (n-1)(1-q)q^{n-2}G(b) + ... + (1-q)^{n-1}G^{n-1}(b)\right)(\overline{\theta}-b)\\ &= \left(q + (1-q)G(b)\right)^{n-1}(\overline{\theta}-b).\end{aligned}$$

Assuming that r is the lower bound of the support of $G(b)$ and that $G(b)$ has no mass points, we get $G(r) = 0$. By indifference, we get for every b in the support:

$$\big(q + (1-q)G(b)\big)^{n-1}(\overline{\theta} - b) = \big(q + (1-q)G(r)\big)^{n-1}(\overline{\theta} - r) = q^{n-1}(\overline{\theta} - r),$$

which immediately gives us

$$G(b) = \frac{q}{1-q}\left[\left(\frac{\overline{\theta} - r}{\overline{\theta} - b}\right)^{\frac{1}{n-1}} - 1\right],$$

and

$$v^*_{\underline{\theta}<r<\overline{\theta}} = (1-q)q^{n-1}(\overline{\theta} - r).$$

To find the upper bound of the support $\overline{b}$ we solve $\frac{q}{1-q}\big[\big(\frac{\overline{\theta}-r}{\overline{\theta}-\overline{b}}\big)^{\frac{1}{n-1}} - 1\big] = 1$ which leads to $\overline{b} = (1-q^{n-1})\overline{\theta} + q^{n-1}r$.

Since only the high-type buyers trade with the seller, the resulting total surplus is given by $(1-q^n)\overline{\theta}$. The resulting revenue of the seller is then given by:

$$\mathcal{R}^*_{\underline{\theta}<r<\overline{\theta}} = (1-q^n)\overline{\theta} - nv^*_{\underline{\theta}<r<\overline{\theta}} = (1-q^n)\overline{\theta} - n(1-q)q^{n-1}(\overline{\theta} - r).$$

□

Case III: $r = \overline{\theta}$

In this case only high types are willing to participate, and they of course have no choice but to bid $b = \overline{\theta}$ in equilibrium, and the resulting revenue will be:

$$\mathcal{R}^*_{r=\overline{\theta}} = (1-q^n)\overline{\theta}.$$

Setting $r > \overline{\theta}$ cannot be an equilibrium strategy. Also, revenue achieved in **Case II** is inferior to that achieved in **Case III**, so setting $\underline{\theta} < r < \overline{\theta}$ cannot be an equilibrium strategy either. The result is then established by directly comparing the revenue in **Case III** to the revenue in **Case I**. □

B Solutions of equilibrium conditions

B.1 Solution of Case 1

Recall that the equilibrium conditions in Case 1 are:

$$v^*_{\text{fse}} = \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{n(1-\delta(1-q)^n)}, \tag{1}$$

$$(1-\delta)\frac{q^{n-1}}{n}(\underline{\theta}-\underline{b}^*) + \delta v^*_{\text{fse}} = 0, \tag{2}$$

$$v^*_{\text{fse}} = \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*) + q^n(\underline{\theta}-\underline{b}^*)\big]. \tag{3}$$

Combining the equations (1) and (2), we get

$$(1-\delta)\frac{q^{n-1}}{n}(\underline{\theta}-\underline{b}^*) + \delta\frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{n(1-\delta(1-q)^n)} = 0,$$

which we can solve for the equilibrium value of $\underline{b}$:

$$\underline{b}^* = \frac{\delta q\big(1-q^n\big)\overline{\theta} + q^n\big(1-\delta(1-q)^n\big)\underline{\theta}}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)},$$

which we can now use to compute the payoff of each type conditional upon winning with $\underline{b}^*$, for a low type buyer we have:

$$\begin{aligned}
\underline{\theta}-\underline{b}^* &= \underline{\theta} - \frac{\delta q\big(1-q^n\big)\overline{\theta} + q^n\big(1-\delta(1-q)^n\big)\underline{\theta}}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)} \\
&= \frac{\delta q\big(1-q^n\big)\underline{\theta} + q^n\big(1-\delta(1-q)^n\big)\underline{\theta} - \delta q\big(1-q^n\big)\overline{\theta} - q^n\big(1-\delta(1-q)^n\big)\underline{\theta}}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)} \\
&= \frac{-\delta q\big(1-q^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)} < 0;
\end{aligned} \tag{4}$$

and for a high type buyer we have:

$$\begin{aligned}
\overline{\theta}-\underline{b}^* &= \overline{\theta} - \frac{\delta q\big(1-q^n\big)\overline{\theta} + q^n\big(1-\delta(1-q)^n\big)\underline{\theta}}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)} \\
&= \frac{\delta q\big(1-q^n\big)\overline{\theta} + q^n\big(1-\delta(1-q)^n\big)\overline{\theta} - \delta q\big(1-q^n\big)\overline{\theta} - q^n\big(1-\delta(1-q)^n\big)\underline{\theta}}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)} \\
&= \frac{q^n\big(1-\delta(1-q)^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big) + q^n\big(1-\delta(1-q)^n\big)} > 0,
\end{aligned} \tag{5}$$

which combined with (1) gives us the resulting equilibrium payoff:

$$\begin{aligned} v^*_{\text{fse}} &= \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{n(1-\delta(1-q)^n)} = \\ &= \frac{(1-\delta)(1-q^n)}{n(1-\delta(1-q)^n)} \times \frac{q^n\big(1-\delta(1-q)^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} \\ &= \frac{1}{n}\frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}. \end{aligned} \tag{6}$$

From (3) we have:

$$\frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\big] = \frac{1}{n}\frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)},$$

which, knowing $\underline{\theta}-\underline{b}^*$ from (4), we can solve for $\overline{\theta}-\overline{b}^*$ to obtain:

$$\begin{aligned} \overline{\theta}-\overline{b}^* &= \frac{(1-\delta)q^n(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} - \frac{q^n(\underline{\theta}-\underline{b}^*)}{1-q^n} \\ &= \frac{(1-\delta)q^n(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} + \frac{q^n\delta q(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} \\ &= \frac{q^n(1-\delta(1-q))(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}, \end{aligned} \tag{7}$$

We can now use expression (7) to determine $\overline{b}^*$:

$$\overline{b}^* = \overline{\theta} - \frac{q^n(1-\delta(1-q))(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}.$$

B.2 Solution of Case 2

Recall that in Case 2 the equilibrium conditions are given by:

$$v^*_{\text{fse}} = \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{n(1-\delta(1-q)^n)}, \tag{8}$$

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)+\delta v^*_{\text{fse}} = (1-\delta)(\overline{\theta}-\overline{b}^*), \tag{9}$$

$$v^*_{\text{fse}} = \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\big]. \tag{10}$$

The equilibrium condition in (8) implies that:

$$(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}) = \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{1-\delta(1-q)^n},$$

which can in turn be rewritten as:

$$(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*) = \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{\theta})}{1-\delta(1-q)^n} + \frac{(1-\delta)(1-q^n)(\underline{\theta}-\underline{b})}{1-\delta(1-q)^n}.$$

Collecting terms, we get:

$$(1-q^n)(\overline{\theta}-\overline{b}^*)+\left[\frac{q^n-\delta q^n(1-q)^n-(1-\delta)(1-q^n)}{1-\delta(1-q)^n}\right](\underline{\theta}-\underline{b}^*)=\frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{\theta})}{1-\delta(1-q)^n}. \tag{11}$$

Recall that (9) implies

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)+\frac{\delta}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b})\big]=(1-\delta)(\overline{\theta}-\overline{b}^*).$$

This condition can be rewritten as:

$$\begin{aligned}
\frac{\delta q^n}{n}(\underline{\theta}-\underline{b}^*)&=(1-\delta)(\overline{\theta}-\overline{b}^*)-(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)-\frac{\delta}{n}(1-q^n)(\overline{\theta}-\overline{b}^*)\\
&=(1-\delta)(\overline{\theta}-\overline{b}^*)-\frac{1-q^n}{n}(\overline{\theta}-\overline{b}^*)\left(\frac{1-\delta}{1-q}+\delta\right)\\
&=(\overline{\theta}-\overline{b}^*)\left[(1-\delta)-\frac{1-q^n}{n(1-q)}(1-\delta q)\right]\\
&=\frac{\big[n(1-q)(1-\delta)-(1-q^n)(1-\delta q)\big](\overline{\theta}-\overline{b}^*)}{n(1-q)}.
\end{aligned} \tag{12}$$

Using equations (11) and (12), we can write:

$$(1-q^n)(\overline{\theta}-\overline{b}^*)+\left[\frac{q^n-\delta q^n(1-q)^n-(1-\delta)(1-q^n)}{1-\delta(1-q)^n}\right](\underline{\theta}-\underline{b}^*)=\frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{\theta})}{1-\delta(1-q)^n},$$

$$\delta q^n(\underline{\theta}-\underline{b}^*)=\frac{\big[n(1-q)(1-\delta)-(1-q^n)(1-\delta q)\big](\overline{\theta}-\overline{b}^*)}{1-q},$$

which can be solved for optimal payoffs $\overline{\theta}-\overline{b}^*$ and $\underline{\theta}-\underline{b}^*$:

$$\begin{aligned}
\underline{\theta}-\underline{b}^*&=-\frac{1}{D(\delta)}\big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big](1-q^n)(\overline{\theta}-\underline{\theta}),\\
\overline{\theta}-\overline{b}^*&=\frac{1}{D(\delta)}\delta q^n(1-q^n)(1-q)(\overline{\theta}-\underline{\theta}),
\end{aligned}$$

where $D(\delta)$ is given by:

$$D(\delta)=q^n\big(1-\delta(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]+(1-q^n)\big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big].$$

The *ex ante* equilibrium payoff can be found from:

$$\begin{aligned}
nv^*_{\text{fse}}&=(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\\
&=\frac{q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{D(\delta)}\big[\delta(1-q^n)(1-q)-(1-q^n)(1-\delta q)+n(1-\delta)(1-q)\big]\\
&=\frac{q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{D(\delta)}\big[(1-q^n)(\delta-\delta q-1+\delta q)+n(1-\delta)(1-q)\big]\\
&=\frac{q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{D(\delta)}(1-\delta)\big[-(1-q^n)+n(1-q)\big].
\end{aligned}$$

Hence the *ex ante* equilibrium payoff is:

$$v^*_{\text{fse}} = \frac{1}{nD(\delta)}(1-\delta)q^n(1-q^n)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}). \tag{13}$$

We can now determine the payoff of the high type who wins with a low bid, i.e. $\overline{\theta}-\underline{b}^*$. Combining the expression for the *ex ante* equilibrium payoff in (13) and the equilibrium condition in (8) we get

$$\frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{n(1-\delta(1-q)^n)} = \frac{1}{nD(\delta)}(1-\delta)q^n(1-q^n)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}),$$

which can be solved for $\overline{\theta}-\underline{b}^*$:

$$\overline{\theta}-\underline{b}^* = \frac{1}{D(\delta)}q^n(1-\delta(1-q)^n)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}).$$

B.3 Solution of Case 3

Recall that in Case 3 the equilibrium conditions are given by:

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)+\delta v^*_{\text{fse}} = (1-\delta)(\overline{\theta}-\overline{b}^*), \tag{14}$$

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)+\delta v^*_{\text{fse}} = (1-\delta)q^{n-1}(\overline{\theta}-\underline{b}^*), \tag{15}$$

$$v^*_{\text{fse}} = \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\big]. \tag{16}$$

Note that conditions (14) and (15) together imply $\overline{\theta}-\overline{b}^* = q^{n-1}(\overline{\theta}-\underline{b}^*)$. Hence the equilibrium payoff becomes:

$$\begin{aligned}
v^*_{\text{fse}} &= \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\big]\\
&= \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\overline{\theta}-\overline{\theta}+\underline{\theta}-\underline{b}^*)\big]\\
&= \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\overline{\theta}-\underline{b}^*)-q^n(\overline{\theta}-\underline{\theta})\big]\\
&= \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q(\overline{\theta}-\overline{b}^*)-q^n(\overline{\theta}-\underline{\theta})\big]\\
&= \frac{1}{n}\big[(1-q^n+q)(\overline{\theta}-\overline{b}^*)-q^n(\overline{\theta}-\underline{\theta})\big].
\end{aligned} \tag{17}$$

The upward incentive compatibility constraint in (14) can then be written as:

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)+\delta\frac{1}{n}\big[(1-q^n+q)(\overline{\theta}-\overline{b}^*)-q^n(\overline{\theta}-\underline{\theta})\big] = (1-\delta)(\overline{\theta}-\overline{b}^*).$$

which can then be solved for $\overline{\theta} - \overline{b}^*$:

$$\overline{\theta} - \overline{b}^* = \frac{\delta q^n (1-q)(\overline{\theta} - \underline{\theta})}{(1-q^n)(1-\delta q) + \delta q(1-q) - n(1-\delta)(1-q)}. \tag{18}$$

We can now introduce shorthand notation for the denominator:

$$D(\delta) = (1-q^n)(1-\delta q) + \delta q(1-q) - n(1-\delta)(1-q).$$

The *ex ante* equilibrium payoff can now be calculated from (17):

$$\begin{aligned}
nv^*_{\text{fse}} &= (1-q^n+q)(\overline{\theta} - \overline{b}^*) - q^n(\overline{\theta} - \underline{\theta}) \\
&= (1-q^n+q)\frac{\delta q^n (1-q)(\overline{\theta} - \underline{\theta})}{(1-q^n)(1-\delta q) + \delta q(1-q) - n(1-\delta)(1-q)} - q^n(\overline{\theta} - \underline{\theta}) \\
&= \frac{q^n(\overline{\theta} - \underline{\theta})}{D(\delta)}\big[(1-q^n+q)\delta(1-q) - (1-q^n)(1-\delta q) - \delta q(1-q) + n(1-\delta)(1-q)\big] \\
&= \frac{q^n(\overline{\theta} - \underline{\theta})}{D(\delta)}\big[(1-q^n)\big(\delta(1-q) - (1-\delta q)\big) + n(1-\delta)(1-q)\big] \\
&= \frac{(1-\delta)q^n(\overline{\theta} - \underline{\theta})}{D(\delta)}\big[n(1-q) - (1-q^n)\big] \\
&= \frac{1}{D(\delta)}(1-\delta)q^n\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}).
\end{aligned}$$

The *ex ante* equilibrium payoff is then given by:

$$v^*_{\text{fse}} = \frac{1}{nD(\delta)}(1-\delta)q^n\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}).$$

The payoff of a high type buyer who wins with the low bid can be calculated from (18) and the fact that $\overline{\theta} - \underline{b}^* = \frac{1}{q^{n-1}}(\overline{\theta} - \overline{b}^*)$, and is therefore given by:

$$\begin{aligned}
\overline{\theta} - \underline{b}^* &= \frac{\delta q(1-q)(\overline{\theta} - \underline{\theta})}{(1-q^n)(1-\delta q) + \delta q(1-q) - n(1-\delta)(1-q)} \\
&= \frac{1}{D(\delta)}\delta q(1-q)(\overline{\theta} - \underline{\theta}).
\end{aligned}$$

A low-type buyer payoff can be calculated from $nv^*_{\text{fse}} = (1-q^n)(\overline{\theta} - \overline{b}^*) + q^n(\underline{\theta} - \underline{b}^*)$:

$$\begin{aligned}
q^n(\underline{\theta} - \underline{b}^*) &= nv^*_{\text{fse}} - (1-q^n)(\overline{\theta} - \overline{b}^*) \\
&= \frac{1}{D(\delta)}(1-\delta)q^n\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}) - (1-q^n)\frac{1}{D(\delta)}\delta q^n(1-q)(\overline{\theta} - \underline{\theta}),
\end{aligned}$$

which implies:

$$\begin{aligned}
\underline{\theta} - \underline{b}^* &= \frac{1}{D(\delta)}\Big[(1-\delta)\big[n(1-q) - (1-q^n)\big] - (1-q^n)\delta(1-q)\Big](\overline{\theta} - \underline{\theta}) \\
&= \frac{1}{D(\delta)}\big[n(1-q)(1-\delta) - (1-q^n)(1-\delta q)\big](\overline{\theta} - \underline{\theta})
\end{aligned}$$

C Some results on the parameter regions

The parameter regions, corresponding to each case, are illustrated by Figure 1:

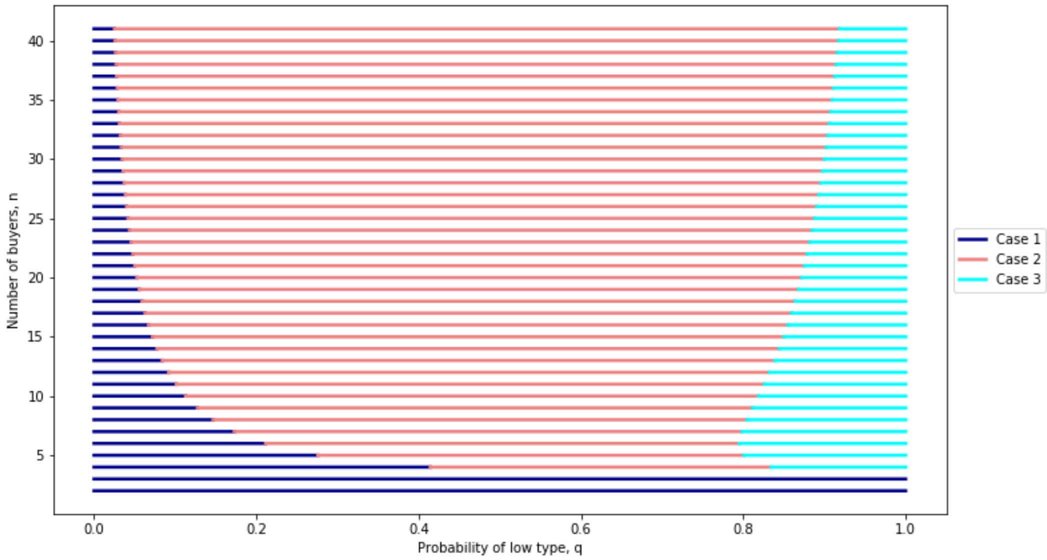

Figure 1: *Parameter regions corresponding to Cases 1, 2, and 3. For each number of buyers n, the respective line shows which values of q belong to Cases 1, 2, and 3.*

C.1 Case 1: High expected valuation/Small number of buyers

The range of parameters, where Case 1 applies, is given by $q < \frac{1-q^n}{n(1-q)}$. It is easy to check that this condition can be satisfied for any q as long as $n = 2$ or $n = 3$, but only for some q if $n \geq 4$. Indeed, consider $n = 2$ first. In this case the condition becomes:

$$2q < \frac{1-q^2}{1-q} \Leftrightarrow 2q < 1 + q \Leftrightarrow q < 1,$$

which is obviously true. If $n = 3$, the condition becomes:

$$3q < \frac{1-q^3}{1-q} \Leftrightarrow 3q < 1 + q + q^2 \Leftrightarrow 0 < 1 - 2q + q^2 \Leftrightarrow 0 < (1-q)^2,$$

which is also obviously true for any $q \in (0, 1)$. If $n = 4$, the condition becomes:

$$4q < \frac{1-q^4}{1-q} \Leftrightarrow 4q < 1 + q + q^2 + q^3 \Leftrightarrow 0 < 1 - 3q + q^2 + q^3$$
$$\Leftrightarrow 0 < (1-q)(-q^2 - 2q + 1) \Leftrightarrow 0 < -q^2 - 2q + 1,$$

which is only true for $q \in (0, -1+\sqrt{2})$. It is possible to establish that for any number of players $n \geq 4$ there will be some values of q falling into Case 1:

Proposition 1. *The equation $1 - q^n = nq(1-q)$ has a unique solution q^* on $(0,1)$ for any $n \geq 4$. Moreover for all $q < q^*$ it is true that $q < \frac{1-q^n}{n(1-q)}$ and vice versa.*

Proof. Both sides of the equation can be divided by $1-q$ to obtain: $\sum_{k=0}^{n-1} q^k - nq = 0$, which can again be divided by $1-q$ to obtain: $1 - \sum_{k=1}^{n-2}(n-1-k)q^k = 0$. Define the function:

$$g(q) \equiv 1 - \sum_{k=1}^{n-2}(n-1-k)q^k.$$

Clearly $g(0) = 1$, and $g(1)$ is given by:

$$\begin{aligned} g(1) &= 1 - \sum_{k=1}^{n-2}(n-1-k) = 1 - (n-1)(n-2) + \sum_{k=1}^{n-2} k \\ &= 1 - (n-1)(n-2) + \frac{(n-1)(n-2)}{2} = 1 - \frac{(n-1)(n-2)}{2} = \frac{n}{2}(3-n) < 0. \end{aligned}$$

hence the equation has a solution on $(0,1)$ for every $n \geq 4$ by the Intermediate Value Theorem.

Consider now the derivative of $g(\cdot)$:

$$g'(q) = -\sum_{k=1}^{n-2}(n-1-k)kq^{k-1} < 0,$$

which implies that the solution q^* is unique and that $q < \frac{1-q^n}{n(1-q)}$ for all $q < q^*$ and vice versa. $\square$

The above proposition essentially shows that for every $n \geq 4$ the restriction divides the interval $(0,1)$ into two parts. In the left part of the segment one will find the values of q that fall into Case 1, and in the right part of the segment one will find the values of q that fall into Cases 2 and 3. Figure 1 provides an illustration and also suggests that, as n goes to infinity, lower and lower values of q fall into Case 1 until there are none left in the limit. Indeed, it is easy to see that $\lim_{n\to\infty} nq(1-q) - (1-q^n) = +\infty$, implying that, for any fixed value of q, the parameter restriction does not hold for all sufficiently high n.

C.2 Case 2: Medium expected valuation

The parameter restrictions of Case 2 in particular imply that:

$$(1-q^n)(1-q) > q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]. \tag{19}$$

In the next proposition I establish that the set of q satisfying (19) is non-empty for any $n \geq 4$ and that there are values q that do not satisfy (19) for every $n \geq 4$.

Proposition 2. *The equation*

$$(1-q^n)(1-q) = q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]$$

has a solution on $(0,1)$ *for every* $n \geq 4$*.*

Proof. Consider the equation:

$$\begin{aligned}
&(1-q^n)(1-q) = q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big] \\
&\Leftrightarrow (1-q^n) = q^{n-1}\big(1-(1-q)^n\big)\left[n-\sum_{k=0}^{n-1} q^k\right] \\
&\Leftrightarrow (1-q)\sum_{k=0}^{n-1} q^k = q^{n-1}\big(1-(1-q)^n\big)(1-q)\sum_{k=0}^{n-2}(n-1-k)q^k \\
&\Leftrightarrow \sum_{k=0}^{n-1} q^k = q^{n-1}\big(1-(1-q)^n\big)\sum_{k=0}^{n-2}(n-1-k)q^k.
\end{aligned}$$

and consider the function:

$$g(q) = q^{n-1}\big(1-(1-q)^n\big)\sum_{k=0}^{n-2}(n-1-k)q^k - \sum_{k=0}^{n-1} q^k.$$

Clearly $g(0) = -1$ and $g(1)$ is computed as:

$$\begin{aligned}
g(1) &= \sum_{k=0}^{n-2}(n-1-k)1^k - \sum_{k=0}^{n-1} 1^k \\
&= (n-1)^2 - \sum_{k=0}^{n-2} k - n \\
&= (n-1)^2 - \frac{(n-1)(n-2)}{2} - n = n\frac{n-3}{2} > 0.
\end{aligned}$$

The result follows by continuity of $g(q)$. $\square$

Recall from Figure 1 that the range of q falling into Case 2 expands as n increases. In the next proposition I establish that any $q \in (0,1)$ will satisfy condition (19) for all sufficiently high values of n:

Proposition 3. *For all $q \in (0,1)$*

$$\lim_{n\to\infty} \left((1-q^n)(1-q) - q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]\right) = 1-q > 0.$$

Proof. Note that the expression can be rewritten as:

$$\underbrace{(1-q^n)(1-q)}_{\to 1-q \text{ as } n\to\infty} - nq^{n-1}\underbrace{(1-q)\big(1-(1-q)^n\big)}_{\to 1-q \text{ as } n\to\infty} + \underbrace{q^{n-1}(1-q^n)\big(1-(1-q)^n\big)}_{\to 0 \text{ as } n\to\infty}.$$

It thus remains to check that $\lim_{n\to\infty} nq^{n-1} = 0$. Taking logs, I get:

$$\begin{aligned}\log\left(nq^{n-1}\right) = \log(n) + (n-1)\log(q) &\leq \sqrt{n-1} + (n-1)\log(q) \\ &= (n-1)\left(\frac{1}{\sqrt{n-1}} + \log(q)\right).\end{aligned}$$

Note that since $\log(q)$ is strictly negative and $\frac{1}{\sqrt{n-1}}$ goes to 0 as n goes to infinity, we have for a large enough n:

$$(n-1)\left(\frac{1}{\sqrt{n-1}} + \log(q)\right) \leq (n-1)\frac{\log(q)}{2}.$$

Since $\log(q) < 0$ we have $\lim_{n\to\infty}(n-1)\frac{\log(q)}{2} = -\infty$, but then $\lim_{n\to\infty}\log\left(nq^{n-1}\right) = -\infty$, which establishes the claim. □

Figure 1 also suggests that the restriction in (19) can be satisfied for all $q \leq \frac{1}{2}$. Indeed, this claim can be shown formally:

Proposition 4. *For all $q \in (0, \frac{1}{2}]$ it is true that*

$$(1-q^n)(1-q) > q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big].$$

Proof. The parameter restriction can be rewritten as:

$$\frac{1-q^n}{1-(1-q)^n} > q^{n-1}\left[n - \sum_{k=0}^{n-1} q^k\right].$$

Observe that $\frac{1-q^n}{1-(1-q)^n} \geq 1$ for all $q \leq \frac{1}{2}$ since $1-q^n \geq 1-(1-q)^n$ is equivalent to $1-q \geq q$. It thus suffices to show that $1 \geq nq^{n-1}$ for all $q \in (0, \frac{1}{2}]$. Define the function $f(q) = nq^{n-1} - 1$. It is clearly strictly increasing in q since $f'(q) = n(n-1)q^{n-2}$. It thus suffices to check that the claim is true for $q = \frac{1}{2}$ or $1 \geq n\frac{1}{2^{n-1}}$ which is equivalent to $2^{n-1} \geq n$, which is true for all $n \geq 2$. □

C.3 Case 3: Low expected valuation

The range of parameters, where Case 3 applies, is defined by:

$$q \geq 1 - \frac{q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]}{1-q^n}.$$

Recall that it in particular implies that $q \geq \frac{1-q^n}{n(1-q)}$ in Case 3. Recall also that $q \geq \frac{1-q^n}{n(1-q)}$ implies that $n \geq 4$ because it cannot be satisfied for any q as long as $n = 2$ or $n = 3$. Combined with the result of Proposition 2, it implies that Case 3 applies to some values of q for all $n \geq 4$, and does not apply to any values of q for $n = 2$ or $n = 3$ (see Figure 1 for an illustration).