Full Surplus Extraction from Colluding Bidders*

Daniil Larionov†

First version: November 4, 2021

This version: July 1, 2023

Click here for the latest version.

Abstract. I consider a repeated auction setting with colluding buyers and a seller who adjusts reserve prices over time without long-term commitment. To model the seller's concern for collusion, I introduce a new equilibrium concept: *collusive public perfect equilibrium*. For every strategy of the seller I define the corresponding "*buyer-game*" in which the seller is replaced by Nature who chooses the reserve prices for the buyers in accordance with the seller's strategy. A public perfect equilibrium is collusive if the buyers cannot achieve a higher symmetric public perfect equilibrium payoff in the corresponding buyer-game. In a setting with symmetric buyers with private binary *iid* valuations and publicly revealed bids, I find *collusive public perfect equilibria* that allow the seller to extract the entire surplus from the buyers in the limit as the buyers' discount factor goes to 1. I therefore show that a non-committed seller can effectively fight collusion even when she faces patient buyers, can only set reserve prices, and has to satisfy stringent public disclosure requirements.

*This paper is based on the first chapter of my PhD dissertation at the University of Mannheim. I thank Thomas Tröger, Takuro Yamashita, and Martin Peitz for their invaluable guidance and support. I also thank João Correia-da-Silva, Vitali Gretschko, Matthias Hunold, Giuseppe Lopomo, Leslie Marx, Alexei Parakhonyak, Patrick Rey, Nicolas Schutz, Anton Sobolev, and the participants of the 2023 MaCCI Annual Conference for useful comments and suggestions. Support by the German Research Foundation (DFG) through CRC TR 224 (Project B01) is gratefully acknowledged.

†ZEW Mannheim. daniil.larionov@zew.de

1 Introduction

Auctions rarely involve a one-shot interaction, often buyers and sellers face each other repeatedly. Procurement decisions for road construction and maintenance, to take one example, have to be made regularly, and public authorities often have to deal with the same pool of potential suppliers. Auctions for electromagnetic spectrum, although less regular, often involve the same pool of potential buyers.

I model a seller who is concerned about collusion among buyers and her own lack of commitment power. I assume that the seller offers an infinite sequence of first-price auctions with adjustable reserve prices and has to satisfy stringent public disclosure requirements: both the reserve prices and the buyers' bids are publicly disclosed after each round of trading. The seller can commit to her chosen reserve price within every period, but does not have enough commitment power to fix the whole dynamic sequence of reserve prices. With respect to collusion, the seller takes a rather pessimistic stance: she expects the buyers to take her chosen strategy as given and try to collectively maximize their own payoff. To model the seller's concern for collusion, I introduce a subclass of strongly symmetric public perfect equilibria, which I call *collusive public perfect equilibria*. For every public strategy of the seller I define the corresponding dynamic game among the buyers (*"buyer-game"*) in which the reserve prices are chosen by Nature in accordance with the seller's strategy; I then select only those equilibria of the repeated first-price auction game, in which the buyers' payoff is no smaller than the maximal payoff they could achieve across all strongly symmetric public perfect equilibria of the corresponding buyer-game. I call the selected public perfect equilibria *collusive*. My main goal is to determine the highest payoff that the seller can obtain in a *collusive public perfect equilibrium* of the repeated first-price auction game.

I consider buyers whose valuations are binary, independent and identically distributed across them and over time. The buyers in my model employ strongly symmetric strategies in any public perfect equilibrium of any buyer-game. In essence, the buyers are prohibited from using more complex asymmetric collusive schemes which might involve communication and/or bidding strategies dependent on each buyer's identity. While it is possible that the seller has less power against a more sophisticated cartel, it should be noted that in practice asymmetric strategies (due to their complexity) might require explicit coordination among the buyers, and explicit coordination could be more easily detected and prevented via the traditional instruments of anti-trust policy. My paper therefore finds a seller's strategy that is robust to collusive schemes that are simpler and more tacit, and thus harder to detect and prove to a court.

I study equilibrium outcomes as the buyers' discount factor goes to 1 and show that collusion in repeated auctions can be dealt with rather effectively: I construct *collusive public perfect equilibria* that achieve full surplus extraction in the limit. These equilibria are stationary along the equilibrium path, feature higher reserve prices than the static outcome, and force the buyers to bid even if their valuation is below the offered reserve price in a given period. Note that, since I am studying a restricted class of public perfect equilibria, my full surplus extraction results do not rely on any of the existing folk theorems. Since these theorems refer to the full set of public equilibrium payoffs, even the mere possibility of full surplus extraction by any *collusive public perfect equilibrium* (let alone by a collusive public perfect equilibrium of any particular structure) is not implied by the existing folk theorems.

To extract full surplus from the buyers, the seller forces the buyer types to separate and punishes any off-equilibrium path deviations she can detect. In the corresponding buyer-game the buyers take the seller's threat as given and might try to deviate to a lower bidding profile. The key to my construction is in identifying the optimal symmetric joint deviation for the buyers and making sure that the construction renders this joint deviation unprofitable. Since the low-type buyers are forced to bid even when their valuation is below the reserve price, the optimal joint deviation will involve the low-type buyers abstaining from participating and receiving the punishment of zero continuation payoffs, and the high-type buyers bidding at the reserve price. Unprofitability of this joint deviation together with incentive compatibility of the buyers' behavior pins down their on-path bidding actions.

Beyond addressing purely theoretical concerns, my results shed light on how collusion can be dealt with in practice. Note that dealing with collusion in repeated first-price auctions is especially challenging because of a fundamental conflict between revenue maximization and fighting collusion. A seller, who wants to maximize her revenue, must force different valuation types of the buyers to separate, making the higher types bid relatively high. But separation creates scope for collusion since, absent punishments, the buyers would try to coordinate on a lower bidding profile. Higher patience will only make this coordination process easier for them. What my results suggest, however, is that the seller can come up with very effective punishments for colluding buyers. To effectively fight collusion, a revenue-maximizing seller should force the buyers to pay "upfront" for the continuation of favorable terms of trade, which is achieved by making the relatively low-valuation types participate even when they have to bid above their current valuations. Penalization of non-participation makes sure that the buyers cannot improve their payoff by making the lower types abstain from the auction altogether and making the higher types take their place in bidding low. Since the higher valuation types also want to avoid (inefficiently) pooling with the lower valuation types, they cannot do anything else but bid high.

1.1 Related literature

The dynamic nature of the interaction presents formidable challenges for an auction designer. Some of those challenges (e.g. intertemporal dependence of agents' private information) have been addressed by the dynamic mechanism design literature (see e.g. Pavan et al. (2014), and Bergemann and Välimäki (2019) for a review). Other important issues however remain. It is well-known that dynamic games often exhibit a multiplicity of equilibria, which makes the classical mechanism design assumption of favorable equilibrium selection harder to justify. For example, in repeated auction settings, collusive outcomes with lower revenue can be supported in equilibrium (see e.g. Skrzypacz and Hopenhayn (2004)). Moreover, collusive equilibria seem to be practically relevant as collusive bidding has been observed in many different repeated auction settings around the world (see e.g. Chassang et al. (2022)).

Repeated auctions are special cases of general repeated games. Equilibria of repeated games were studied by Abreu et al. (1990), who provide a recursive characterization of equilibrium payoffs for repeated games with imperfect monitoring, and Fudenberg et al. (1994) who prove a folk theorem for these games. Athey et al. (2004) introduce (*iid*) private information into a repeated Bertrand game with imperfect monitoring and apply the recursive characterization of Abreu et al. (1990) to their game. They show that patient players can sustain high rigid prices in the optimal equilibrium, thus extracting a lot of surplus from the consumers. Their model can be translated to an auction setting with a passive seller who chooses a reserve price once and for all in the beginning of the game. In the buyer-optimal equilibrium with patient buyers such a seller would be forced to sell the good at her chosen reserve price in every period.

Even though the literature on collusion in repeated auctions and oligopolies with private information is very extensive (see Correia-da-Silva (2017) for a review), very few papers are concerned with the study of how the seller's behavior might affect the buyers' collusion. Abdulkadiroglu and Chung (2004) consider a stage game design problem in which a committed seller proposes a mechanism that will become the stage game played repeatedly by tacitly colluding buyers. The seller in their model is concerned with buyers coordinating on the buyer-optimal sequential equilibrium and designs the stage game accordingly. Similarly to my paper, Abdulkadiroglu and Chung (2004) find a mechanism that extracts the entire surplus from the buyers. In their mechanism, all the buyers pay the same participation fee to the seller and then the partnership dissolution mechanism of Cramton et al. (1987) is run. Abdulkadiroglu and Chung (2004) however note that a non-committed seller will fall far short of full surplus extraction: in the buyer-optimal sequential equilibrium of the repeated game, in which the seller moves first and proposes a mechanism, the seller's revenue will be zero. In my paper, I propose a less pessimistic (from the seller's point of view) equilibrium selection model. While the seller in my model lacks long-term commitment, she is able to control her own strategy and does not have to coordinate on the worst equilibrium for herself. She cannot however guarantee that the buyers will coordinate on her preferred equilibrium either. The buyers could take her strategy as given and tacitly coordinate on a lower bidding profile, hence the seller's equilibrium strategy must make such coordination unprofitable for the buyers. Although the seller has a more active role in equilibrium selection in my model, she is more constrained in terms of feasible mechanisms: she must offer a first-price auction in every period and can only adjust reserve prices over time.

Thomas (2005) notices that a seller could make collusion harder for the buyers by raising reserve prices, but assumes that the seller moves only once, in the beginning of time, and chooses one reserve price for the entirety of the repeated game between the buyers. Zhang (2022) studies a class of collusive agreements between buyers in a model of repeated first-price auctions, and, as a side note to his main results, shows how a revenue-maximizing seller should respond to collusion. His seller, much like the seller in Thomas (2005), moves only once and commits to a single reserve price. As the discount factor goes to 1, his seller is forced to accommodate "full collusion" with all the bids being suppressed down to the reserve price, which results in a revenue far short of full surplus. Iossa et al. (2023) study collusion mitigation strategies in the context of an infinite-horizon model with two suppliers and two buyers who simultaneously offer (reverse) first-price auctions to the buyers. Along with other mitigation strategies, Iossa et al. (2023) consider the use of reserve prices to deter collusion among the suppliers. Like in Thomas (2005) and Zhang (2022), their buyers commit to a single reserve price in the beginning of the game, and therefore also obtain far less than full surplus when the discount factor is close to 1.

Ortner et al. (2022) are also concerned with mitigating the effects of collusion in repeated reverse auctions. They propose a model with a regulator who observes the whole (infinite) bidding history and can punish colluding bidders. They construct tests for detecting collusive patterns of behavior which only allow for false negatives – therefore competitive bidders pass them with probability one. The regulator can then use the outcomes of the tests to punish the colluding bidders. In contrast to their work, my seller only has access to finite histories of bids and can only use reserve prices to punish the buyers.

Bergemann and Hörner (2018) study a binary type model of first-price auctions similar to mine. The seller in their model is however passive and does not set a reserve price at all, and the buyers' valuations are perfectly persistent. They look at disclosure regimes regarding the bidding and the winning histories. In contrast to the findings in my paper, they show that the maximal disclosure regime leads to inefficient equilibria with low revenues. I show that an active seller who can adjust reserve prices over time can extract full surplus even when the full history of bids and identities of the winning buyers is publicly disclosed.

My paper is also related to the literature on collusion in static auctions. This literature was initiated by McAfee and McMillan (1992), who study the outcomes of explicit before-auction communication in a first-price auction setting. They solve for optimal collusive schemes with ("strong collusion") and without transfers ("weak collusion"). In the optimal weak collusion scheme, the bidders bid at the reserve price as long as their valuation exceeds it and abstain otherwise. In the optimal strong collusion scheme, the colluding buyers can obtain a higher expected payoff by running a "knock-out" auction among themselves. The winner of the knock-out auction bids at the reserve price (as long as it exceeds his valuation) in the legitimate auction, and the losers are compensated for abstaining from the legitimate auction. It is however known now, that in the static setting the seller can avoid the dramatic losses from collusion via more sophisticated auction design. Che and Kim (2009) show that the second-best auction can be made collusion-proof, even when the bidders can use transfers to collude.

Finally, my paper speaks to the large literature on robustness in mechanism design (see Carroll (2019) for a comprehensive review). In my paper the seller aims to be robust to collusive behavior of the buyers.

1.2 Roadmap

The rest of the paper is organized as follows: Section 2 introduces the model of a repeated first-price auction game. In Section 3, I introduce the definitions of a *buyer-game* and a *collusive public perfect equilibrium*. In Section 4, I state the main result of the paper by explicitly constructing collusive public perfect equilibria that allow the seller to extract full surplus in the limit. Sections 5, 6, and 7 are devoted to proving the main result. Section 8 briefly discusses the optimal reserve prices. Finally, Section 9 concludes.

2 Model

2.1 Setup

There is a seller (player 0) and $n \geq 2$ buyers (players $1, \ldots, n$) who interact over infinitely many periods. The seller sells one unit of a private good in every period via a first-price auction with a reserve price. Each buyer is privately informed about his valuation type, which is drawn from a binary set $\Theta = \{\underline{\theta}, \overline{\theta}\}$, with $0 \leq \underline{\theta} < \overline{\theta}$, *iid* across periods and buyers. The probability of the low type $\underline{\theta}$ is $q \in (0, 1)$. The buyers share a common discount factor $\delta \in (0, 1)$. The seller's discount factor is $\delta_0 \in (0, 1)$. The timing of each period is as follows:

1. Seller announces a reserve price r.
2. Buyers privately learn their valuations for the good in the current period.
3. Buyers bid or abstain ($\emptyset$) in the first-price auction with the reserve price r.
4. The winner (if any) is determined, the buyers' choices are publicly disclosed.

The action set of the seller is $A_0 = \mathbb{R}_+$, the action set of each buyer is $A = \{\emptyset\} \cup \mathbb{R}_+$.

Buyer i's payoff is equal to his valuation θ_i net of his bid b_i if he wins the auction, and zero otherwise. Ties are broken uniformly. Formally,

$$u_i(r,b,\theta_i) = \begin{cases} \frac{1}{\#(\text{win})}(\theta_i - b_i), & \text{if } b_i \geq r \;\&\; \big(b_i = \max\{b_1,...,b_n\} \text{ or } b_{-i} = \emptyset\big) \\ 0, & \text{otherwise} \end{cases},$$

where #(win) stands for the number of winners in the auction, i.e. the number of buyers who placed the highest bid. The seller's revenue is equal to the highest bid if there is a buyer who bids above his reserve price, and zero otherwise:

$$\mathcal{R}(r,b) = \begin{cases} b_i, & \text{if } b_i \geq r \;\&\; \big(b_i = \max\{b_1,...,b_n\} \text{ or } b_{-i} = \emptyset\big) \\ 0, & \text{otherwise} \end{cases}.$$

2.2 One-shot auctions

Before we turn our attention to the repeated auction problem, let us consider equilibria of the stage game. The intuition here is straightforward. If there are relatively few low types in the population (small q), the seller will prefer to trade with high types only, and will therefore set the reserve price to $\overline{\theta}$. The low-type buyers will abstain while the high-type buyers will bid their valuation $\overline{\theta}$. If there are relatively many low types in the population (large q), the seller will prefer to trade with both types, and will therefore set the reserve price to $\underline{\theta}$. The low-type buyers will bid their valuation while the high-type buyers will play a mixed strategy whose support lies above $\underline{\theta}$. Proposition 0 characterizes equilibrium payoffs:

Proposition 0. One-shot auction equilibrium payoffs

- *If the parameters of the model fall into the* ***High-reserve-price region*** $\big(q < \frac{n(\overline{\theta}-\underline{\theta})}{\underline{\theta}+n(\overline{\theta}-\underline{\theta})}\big)$, *then the seller sets* $r^*_{os} = \overline{\theta}$ *and generates revenue* $\mathcal{R}^*_{os} = (1-q^n)\overline{\theta}$*; the buyers get the ex ante payoff* $v^*_{os} = 0$.
- *If the parameters of the model fall into the* ***Low-reserve-price region*** $\big(q \geq \frac{n(\overline{\theta}-\underline{\theta})}{\underline{\theta}+n(\overline{\theta}-\underline{\theta})}\big)$, *then the seller sets* $r^*_{os} = \underline{\theta}$ *and generates revenue* $\mathcal{R}^*_{os} = (1-q^n)\overline{\theta}+q^n\underline{\theta}-n(1-q)q^{n-1}(\overline{\theta}-\underline{\theta})$*; the buyers get the ex ante payoff* $v^*_{os} = (1-q)q^{n-1}(\overline{\theta}-\underline{\theta})$.

Proof. See Appendix A. □

3 Collusive Public Perfect Equilibrium

3.1 Motivation

Let us consider the Low-reserve-price region and the infinite repetition of the associated one-shot equilibrium. Clearly, it is an equilibrium of the infinitely repeated auction game, but there is no reason to believe that the players will actually coordinate on it. In fact, the possibility of buyers' collusion provides a good reason to believe otherwise. Suppose the buyers, instead of coordinating on their one-shot equilibrium strategies, use a different bidding profile, in which any high-type buyer bids $\overline{b} = \underline{\theta}$ and any low-type buyer abstains in every period. The new bidding profile gives a lower revenue of $(1 - q^n)\underline{\theta}$ to the seller and a higher payoff of $\frac{1}{n}(1 - q^n)(\overline{\theta} - \underline{\theta})$ to the buyers. The buyers can support their new bidding profile using a "grim-trigger" strategy, which punishes deviations by moving back to the one-shot equilibrium strategies of the Low-reserve-price region; the buyers only have to make sure that the high types do not want to deviate to $\underline{\theta} + \epsilon$, i.e. that

$$\underbrace{(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\underline{\theta}) + \delta\frac{1}{n}(1-q^n)(\overline{\theta}-\underline{\theta})}_{\text{Payoff of a high type from } \overline{b}=\underline{\theta},\ \underline{b}=\emptyset} \geq \underbrace{(1-\delta)(\overline{\theta}-\underline{\theta})}_{\text{Today's deviation payoff}} + \underbrace{\delta(1-q)q^{n-1}(\overline{\theta}-\underline{\theta})}_{\text{Grim punishment payoff}},$$

which can be satisfied for high enough values of δ.

As we can see, the infinite repetition of the one-shot equilibrium in the Low-reserve-price region is not *"collusive"* in the sense that the buyers do not exploit their ability to collude to the fullest extent possible. A seller who has concerns about buyers' collusion should not hope to end up in such an equilibrium and needs to consider more sophisticated strategies. The seller's equilibrium strategy should however always guarantee that the buyers cannot improve their payoff similarly to how they did it in the above example. I formalize this requirement by introducing the concept of *collusive public perfect equilibrium.*

3.2 Definition

A *collusive public perfect equilibrium* is a strongly symmetric public perfect equilibrium that satisfies two novel requirements:

1. **Collusiveness on path.** The buyers must collude given the seller's on-path play of her equilibrium strategy. Central to this requirement is the notion of a *buyer-game* I

formally introduce below in Definition 3. The buyer-game corresponding to a seller's strategy is a stochastic first-price auction game between the buyers, in which the reserve prices are determined according to the seller's strategy. Collusiveness on path requires that the buyers be unable to improve their payoff by moving to a different strongly symmetric public perfect equilibrium in the buyer-game induced by the seller's equilibrium strategy. In the above example of the infinite repetition of the Low-reserve-price region equilibrium collusiveness on path was violated since the buyers could improve their payoff by moving to a different equilibrium between themselves. See Subsection 3.2.1 for a formal definition of collusiveness on path.

2. **Collusiveness off path.** As long as the buyers stick to their equilibrium strategies, the continuation play must be *collusive on path* (in the sense of the first requirement) regardless of seller's actions. Colusiveness off path formalizes the idea that buyers' collusive agreements cannot be broken by seller's actions. It does however allow non-collusive equilibria to be played following buyers' deviations and thus imposes no restriction on the buyers' ability to punish deviators from their collusive agreements. See Subsection 3.2.2 for a formal definition of collusiveness off path.

Strongly symmetric public perfect equilibrium is a public perfect equilibrium, in which buyers take symmetric actions on and off the equilibrium path. Public perfect equilibrium is an equilibrium in *public strategies*, i.e. strategies which map *public histories* into players' actions. A *public history* in the beginning of period $t+1$ is a sequence that includes all the actions taken by each player up to that period: $\big(\varnothing, (r_0, b_{10}, \dots, b_{n0}), \dots, (r_t, b_{1t}, \dots, b_{nt})\big)$, where $\varnothing$ denotes the initial history. The set of those histories is $\mathcal{H}_0 \equiv \cup_{t=0}^{\infty}\big(A_0 \times A^n\big)^t$, with a typical period-$t$ history denoted h_0^t. Since the buyers additionally observe the action taken by the seller in every period, the set of public histories at which they get to make a move is $\mathcal{H} \equiv \cup_{t=0}^{\infty}\big[\big(A_0 \times A^n\big)^t \times A_0\big]$ with a typical period-t history denoted h^t. A pure public strategy for the seller is a mapping $\sigma_0 : \mathcal{H}_0 \to A_0$, for the buyers it is $\sigma_i : \mathcal{H} \times \Theta \to A$.

The expected payoff of the seller in the repeated auction game is given by:

$$U_0(\sigma) = (1-\delta_0)\,\mathbb{E}\sum_{t=0}^{\infty} \delta_0^t \mathcal{R}\big(\sigma_0(h_0^t), \sigma_i(h^t, \theta_{it}), \sigma_{-i}(h^t, \theta_{-it})\big).$$

The expected payoff of the buyers $i = 1, 2, ..., n$ in the repeated auction game is given by:

$$U_i(\sigma) = (1-\delta)\,\mathbb{E}\sum_{t=0}^{\infty}\delta^t u_i\big(\sigma_0(h_0^t), \sigma_i(h^t, \theta_{it}), \sigma_{-i}(h^t, \theta_{-it}), \theta_{it}\big).$$

The above definitions extend naturally to behavioral strategies. We can now state the formal definition:

Definition 1. Strongly symmetric public perfect equilibrium

A strategy profile $(\sigma_0^*, \sigma_1^*, ..., \sigma_n^*)$ is a strongly symmetric public perfect equilibrium if

1. *it induces a Nash equilibrium after every public history $h_0 \in \mathcal{H}_0$ and $h \in \mathcal{H}$;*
2. *$\sigma_i^*(h, \theta) = \sigma_j^*(h, \theta)$ after any public history $h \in \mathcal{H}$ for any buyers i, j and any θ.*

The first condition of Definition 1 rules out non-credible threats at every public history much like subgame perfect equilibrium rules out non-credible threats in every subgame. The second condition makes sure that the buyers use symmetric bidding actions on and off the equilibrium path. Note that strongly symmetric public perfect equilibria have recursive structure: the continuation play after any public history is itself a strongly symmetric public perfect equilibrium.

All strongly symmetric public perfect equilibria I construct below, except the infinite repetition of the one-shot equilibrium in the Low-reserve-price region, satisfy the following additional assumption:

Assumption 1(a). Pure bidding actions on path

Buyers use pure bidding actions on path, i.e. after any public history $h \in \mathcal{H}$ consistent with the on-path play of $(\sigma_0^*, \sigma^*, \ldots, \sigma^*)$, the action $\sigma^*(h, \theta)$ is pure for both $\theta \in \{\underline{\theta}, \overline{\theta}\}$.

Assumption 1(a) itself is not restrictive since we can construct a full-surplus-extracting strongly symmetric public perfect equilibrium that belongs to the class of equilibria allowed by Assumption 1(a). However, I make a similar assumption in the next subsection (Assumption 1(b)), which forces the buyers to play the same class of equilibria in any buyer-game, restricting the set of collusive schemes they could use. It remains an open question whether Assumptions 1(a) and 1(b) could be dispensed with.

3.2.1 Collusiveness on path

To define *collusiveness on path* formally, we have to introduce the notion of the *buyer-game* induced by a seller's strategy. To define the states in the buyer-game, we first define the *path automaton* of a seller's strategy[1]. In order to do that, fix a particular pure public strategy[2] of the seller σ_0. Let $\tilde{\mathcal{H}}_0(\sigma_0)$ be the set of histories consistent with the seller's play of σ_0 and any profile of buyers' strategies[3]. Two histories h_0 and h_0' from $\tilde{\mathcal{H}}_0(\sigma_0)$ are called σ_0-equivalent if they prescribe the same continuation play for the seller according to σ_0, i.e. if $\sigma_0|_{h_0} = \sigma_0|_{h_0'}$. Let Ω be the resulting set of equivalence classes with ω^0 being the equivalence class of the initial history $\varnothing$. The path automaton of σ_0 is defined as follows:

Definition 2. Path automaton of a seller's strategy

The path automaton of σ_0 is the tuple $\left(\Omega, \omega^0, r, \tau\right)$, where

- *$r : \Omega \to A_0$ is the decision rule satisfying $r(\omega) = \sigma_0(h_0)$ for any $h_0 \in \omega$.*
- *$\tau : \Omega \times A^n \to \Omega$ is the transition function satisfying $\tau(\omega, b) = w'$ if and only if for any history $h_0 \in w$ the concatenated history $(h_0, r(\omega), b) \in w'$.*

We can now introduce the formal definition of the buyer-game induced by σ_0:

Definition 3. Buyer-game

Let $\left(\Omega, \omega^0, r, \tau\right)$ be the path automaton of σ_0. The buyer-game induced by σ_0 is a stochastic game between the buyers, where:

- *The set of actions for each buyer is A, i.e. is as defined in the repeated auction game.*
- *The set of states is Ω, with the initial state ω^0. State transitions occur according to τ.*
- *The set of valuations is Θ, i.e. is as defined in the repeated auction game.*
- *The utility of buyer i with type θ_i bidding b_i in state ω is*

$$\tilde{u}_i(\omega, b, \theta_i) = \begin{cases} \frac{1}{\#(\text{win})}(\theta_i - b_i), & \text{if } b_i \geq r(\omega) \text{ \& } \left(b_i = \max\{b_1, ..., b_n\} \text{ or } b_{-i} = \emptyset\right) \\ 0, & \text{otherwise} \end{cases},$$

[1] Unlike an automaton representation, the path automaton of a seller's strategy assumes that the seller never deviates from σ_0, and therefore represents only part of her repeated game strategy. See also Kandori and Obara (2006) who employ a similar definition in the context of repeated games with private monitoring.

[2] It is w.l.o.g. to restrict attention to pure strategies of the seller, since our goal is to construct a full-surplus-extracting collusive public perfect equilibrium, which can be achieved under this restriction.

[3] A typical element of $\tilde{\mathcal{H}}_0(\sigma_0)$ can be written as $h_0^t = \big(\varnothing,\ (\sigma_0(\varnothing),\ b_0),\ (\sigma_0(h_0^1),\ b_1),\ \dots,\ (\sigma_0(h_0^{t-1}),\ b_{t-1})\big)$; where $h_0^1 = (\sigma_0(\varnothing),\ b_0)$, $h_0^2 = \big((\sigma_0(\varnothing),\ b_0),\ (\sigma_0(h_0^0),\ b_1)\big)$, etc.

where $\#(\mathrm{win})$ *is the number of winners in the auction.*

Let us look at the strongly symmetric public perfect equilibria of the buyer-game induced by σ_0. A public history at period $t+1$ in the buyer-game includes all states and bids up to period $t+1$: $\big(\omega_0, (b_{10}, \ldots, b_{n0}), \ldots, \omega_t, (b_{1t}, \ldots, b_{nt}), \omega_{t+1}\big)$. Let $\mathbf{H}(\sigma_0)$ be the set of these public histories. A pure public strategy in the buyer game is a function $\rho_i : \mathbf{H}(\sigma_0) \times \Theta \to A$. This definition extends naturally to behavioral strategies. A strongly symmetric public perfect equilibrium of the buyer-game induced by a seller's strategy σ_0 is defined as follows:

Definition 4. Strongly symmetric public perfect equilibrium of a buyer-game

A strategy profile $(\rho_1^*, \ldots, \rho_n^*)$ *is a strongly symmetric public perfect equilibrium equilibrium of the buyer-game induced by* σ_0 *if*

1. *it induces a Nash equilibrium after any public history* $\mathbf{h} \in \mathbf{H}(\sigma_0)$*;*
2. $\rho_i^*(\mathbf{h}, \theta) = \rho_j^*(\mathbf{h}, \theta)$ *after any public history* $\mathbf{h} \in \mathbf{H}(\sigma_0)$ *for any buyers* i, j *and any* θ*.*

Recall that by Assumption 1(a) the buyers use pure bidding actions on path of any strongly symmetric public perfect equilibrium of the repeated auction game. Assumption 1(b) restricts the buyers to play equilibria from the same class in any buyer-game:

Assumption 1(b). Pure bidding actions on path of a buyer-game

Buyers use pure bidding actions on path in the buyer-game induced by σ_0*, i.e. after any public history* $\mathbf{h} \in \mathbf{H}(\sigma_0)$ *consistent with the on-path play of* $(\rho^*, \ldots, \rho^*)$*, the action* $\rho^*(\mathbf{h}, \theta)$ *is pure for both* $\theta \in \{\underline{\theta}, \overline{\theta}\}$*.*

Assumption 1(b) does not allow the buyers to collude by moving to a strongly symmetric public perfect equilibrium of the buyer game that involves using mixed actions along the equilibrium path. This assumption may be restrictive as it could in principle be possible that the buyers could collectively benefit from using mixed actions in the buyer-games induced by the full-surplus-extracting collusive equilibria constructed in Section 4 below. It can however be shown that the simplest collusive schemes with mixed actions do not help the buyers to improve their payoff[4]. The bigger question of whether Assumption 1(b) could be dispensed with remains open.

We can now use the above definitions to formally introduce *collusiveness on path*:

[4]For example, some stationary schemes, in which the high types mix over two bidding actions on path, do not improve the buyers' payoff because of their efficiency loss vis-à-vis fully separating behavior.

Definition 5. Collusiveness on path

A strongly symmetric public perfect equilibrium $(\sigma_0^*, \sigma^*, ..., \sigma^*)$ *of the repeated auction game is collusive on path if there is no strongly symmetric public perfect equilibrium with pure actions along the equilibrium path (i.e. satisfying Assumption 1(b)) in the buyer-game induced by* σ_0^**, whose equilibrium payoff exceeds the buyers' payoff from* $(\sigma_0^*, \sigma^*, ..., \sigma^*)$ *in the repeated auction game.*

3.2.2 Collusiveness off path

Recall that the requirement of *collusiveness off path* formalizes the idea that buyers' collusive agreements cannot be broken by seller's actions. More specifically, if the buyers have played their equilibrium actions up to a given period, then they must keep colluding on path no matter what the seller has played. Here is the formal definition:

Definition 6. Collusiveness off path

Suppose $(\sigma_0^*, \sigma^*, \ldots, \sigma^*)$ *is a strongly symmetric public perfect equilibrium of the repeated auction game. Consider an alternative seller's strategy* σ_0' *and let* $h_0^t \in \mathcal{H}_0$ *be a period-t history consistent with the on-path play of* $(\sigma_0', \sigma^*, \ldots, \sigma^*)$*. If the continuation equilibrium* $(\sigma_0^*|_{h_0^t}, \sigma^*|_{h_0^t}, \ldots, \sigma^*|_{h_0^t})$ *is collusive on path for any such* h_0^t *and* σ_0'*, then* $(\sigma_0^*, \sigma^*, \ldots, \sigma^*)$ *is collusive off path.*

Summarizing the above, I can state the main definition formally:

Definition 7. Collusive public perfect equilibrium

A strongly symmetric public perfect equilibrium of the repeated auction game is a collusive public perfect equilibrium if it is collusive on and off path.

Remark 1. *Observe that the infinite repetition of the one-shot equilibrium in the High-reserve-price region is a collusive public perfect equilibrium. It is clearly a strongly symmetric public perfect equilibrium. To show collusiveness on path, observe that the buyers get zero along the equilibrium path, and it is not possible to improve their payoff once the seller's on-path play is fixed: bidding below* $\overline{\theta}$ *leads to a zero payoff as well, bidding above* $\overline{\theta}$ *can only lead to losses. After a deviation by any player the play returns to the same equilibrium in the next period, hence it is also collusive off path.*

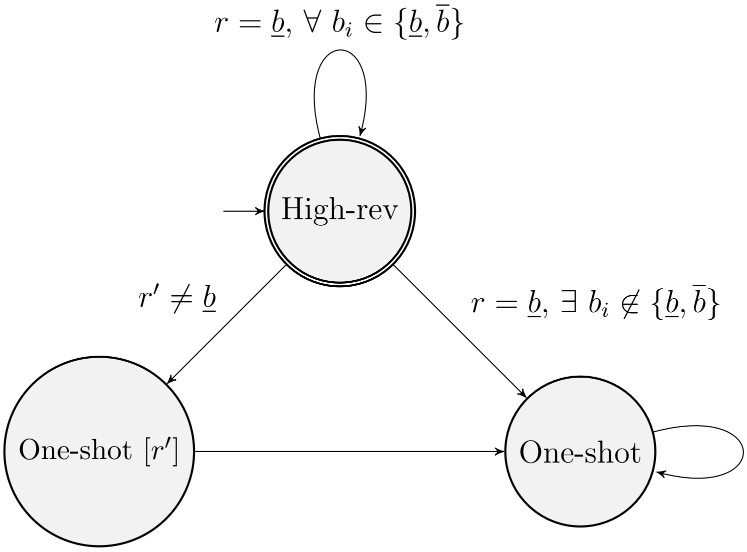


Figure 1: High-revenue strategy profile in the High-reserve-price region.

4 Main result: full surplus extraction

I will now explicitly introduce a class of strategy profiles that allow the seller to extract full surplus in the limit as the buyers' discount factor δ goes to 1. These strategy profiles are stationary and separating on path. Their full formal description is given by Definition 8.

Definition 8. High-revenue strategy profiles

Fix a pair of bids $(\bar{b}, \underline{b})$*. To define the* ***high-revenue strategy profiles*** *corresponding to* $(\bar{b}, \underline{b})$*, we distinguish three cases. In all three cases, the on-path behavior is the same: the game starts in the High-revenue state* **(High-rev)***. In* **(High-rev)** *any low-type buyer bids* $\underline{b}$ *and any high-type buyer bids* $\bar{b}$*, and the seller sets the reserve price equal to the bid of a low-type buyer,* $r = \underline{b}$*. As long as the seller sets* $r = \underline{b}$ *and no buyer deviates to a bid outside* $\{\underline{b}, \bar{b}\}$*, the play remains in* **(High-rev)***.*

Off path, the following three definitions apply, depending on the parameter values:

(i) In the High-reserve-price region, there is also a continuum of states **(One-shot** $[r']$**)** *with one state for each* $r' \in [0, +\infty)$*, and the* **(One-shot)***-state. The following applies:*

- *If, in* **(High-rev)***, the seller deviates to a reserve price* $r' \neq \underline{b}$*, then the play switches to* **(One-shot** $[r']$**)***. If, in* **(High-rev)***, a buyer deviates to a bid outside of* $\{\underline{b}, \bar{b}\}$*, the play switches to* **(One-shot)***.*

- *In any* **(One-shot** $[r']$**)**, *the buyers play the equilibrium of the one-shot auction with the reserve price* r' *once, the play then switches to* **(One-shot)**.
- *In* **(One-shot)**, *the one-shot equilibrium of the High-reserve-price region is infinitely repeated.*

This case is illustrated by Figure 1.

(ii) In the Low-reserve-price region with $q \geq \frac{\underline{\theta}}{\overline{\theta}-\underline{\theta}}$, *there are four additional states: the high-reserve-price state* **(HRP)**, *the abstention state* **(Abstain)**, *the low-revenue-separating state* **(LRS)**, *and the* **(One-shot)**-*state. The following applies:*

- *If, in* **(High-rev)**, *the seller deviates to a reserve price* $r' \neq \underline{b}$, *then the play switches to* **(Abstain)**. *If, in* **(High-rev)**, *a buyer deviates to a bid outside of* $\{\underline{b}, \overline{b}\}$, *then the play switches to* **(HRP)**.
- *In* **(HRP)**, *the seller sets* $r_{\text{hrp}} = \overline{\theta}$, *any low-type buyer abstains, and any high-type buyer bids* $\overline{b}_{\text{hrp}} = \overline{\theta}$. *As long as the seller plays* $r_{\text{hrp}} = \overline{\theta}$, *the play remains in* **(HRP)**. *If, in* **(HRP)**, *the seller deviates to a reserve price* $r' \neq \overline{\theta}$, *then the play switches to* **(Abstain)**.
- *In* **(Abstain)**, *the buyers of both types abstain, and afterwards the play immediately switches to* **(LRS)**. *If, in* **(Abstain)**, *a buyer deviates and places a bid, the play switches to* **(One-shot)**.
- *In* **(LRS)**, *the seller sets* $r_{\text{lrs}} = 0$, *any low-type buyer bids* $\underline{b}_{\text{lrs}} = 0$, *and any high-type buyer bids* $\overline{b}_{\text{lrs}} = \frac{1-q^{n-1}}{1-q^n}\underline{\theta}$. *As long as the seller sets* $r_{\text{lrs}} = 0$ *and no buyer deviates to a bid outside of* $\{\underline{b}_{\text{lrs}}, \overline{b}_{\text{lrs}}\}$, *the play remains in* **(LRS)**. *If, in* **(LRS)**, *the seller deviates to a reserve price* $r' \neq 0$, *then the play switches to* **(Abstain)**. *If, in* **(LRS)**, *a buyer deviates to a bid outside of* $\{\underline{b}_{\text{lrs}}, \overline{b}_{\text{lrs}}\}$, *the play switches to* **(One-shot)**.
- *In* **(One-shot)**, *the one-shot equilibrium of the Low-reserve-price region is infinitely repeated.*

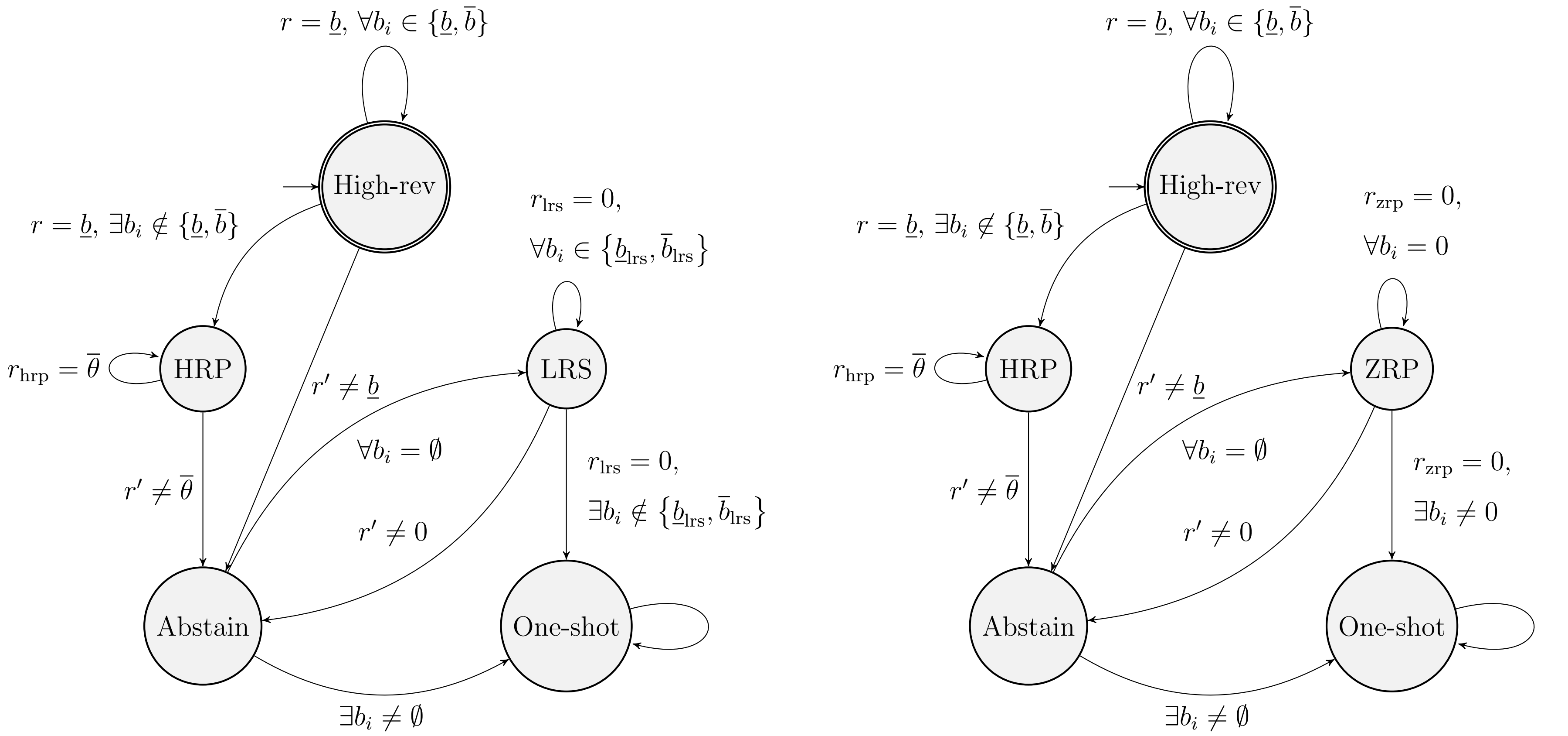
This case is illustrated by Figure 2a.

(iii) In the Low-reserve-price region with $q < \frac{\underline{\theta}}{\overline{\theta}-\underline{\theta}}$, *there are four additional states: the high-reserve-price state* **(HRP)**, *the abstention state* **(Abstain)**, *the zero-revenue-pooling*

state **(ZRP)**, *and the* **(One-shot)**-*state. The following applies:*

- *If, in* **(High-rev)**, *the seller deviates to a reserve price* $r' \neq \underline{b}$, *then the play switches to* **(Abstain)**. *If, in* **(High-rev)**, *a buyer deviates to a bid outside of* $\{\underline{b}, \overline{b}\}$, *then the play switches to* **(HRP)**.
- *In* **(HRP)**, *the seller sets* $r_{\text{hrp}} = \overline{\theta}$, *any low-type buyer abstains, and any high-type buyer bids* $\overline{b}_{\text{hrp}} = \overline{\theta}$. *As long as the seller plays* $r_{\text{hrp}} = \overline{\theta}$, *the play remains in* **(HRP)**. *If, in* **(HRP)**, *the seller deviates to a reserve price* $r' \neq \overline{\theta}$, *then the play switches to* **(Abstain)**.
- *In* **(Abstain)**, *the buyers of both types abstain, and afterwards the play immediately switches to* **(ZRP)**. *If, in* **(Abstain)**, *a buyer deviates and places a bid, the play switches to* **(One-shot)**.
- *In* **(ZRP)**, *the seller sets* $r_{\text{zrp}} = 0$, *and both buyer types pool at zero, i.e.* $\underline{b}_{\text{zrp}} = \overline{b}_{\text{zrp}} = 0$. *As long as the seller sets* $r_{\text{zrp}} = 0$ *and all buyers bid zero, the play remains in* **(ZRP)**. *If, in* **(ZRP)**, *the seller deviates to a reserve price* $r' \neq 0$, *then the play switches to* **(Abstain)**. *If, in* **(ZRP)**, *a buyer deviates to a non-zero bid, the play switches to* **(One-shot)**.
- *In* **(One-shot)**, *the one-shot equilibrium of the Low-reserve-price region is infinitely repeated.*

This case is illustrated by Figure 2b.



(a) Low-reserve-price region and $q \geq \frac{\underline{\theta}}{\overline{\theta} - \underline{\theta}}$.

(b) Low-reserve-price region and $q < \frac{\underline{\theta}}{\overline{\theta} - \underline{\theta}}$.

Figure 2: High-revenue strategy profiles in the Low-reserve-price region.

To formulate the main result, I first set up the following *revenue-maximization problem*:

$$\mathcal{RM}: \ \mathcal{R}^*_{\text{fse}}(\delta) \equiv \max_{\bar{b},\underline{b},v}(1-q^n)\bar{b} + q^n\underline{b}, \quad \text{s.t.}$$

$$\text{(Eq-payoff)} \qquad v = \frac{1}{n}\big[(1-q^n)(\bar{\theta}-\bar{b}) + q^n(\underline{\theta}-\underline{b})\big];$$

Incentive constraints:

$$\text{(LowIC)} \qquad (1-\delta)\frac{q^{n-1}}{n}(\underline{\theta}-\underline{b}) + \delta v \geq 0,$$

$$\text{(HighIC-up)} \qquad (1-\delta)\frac{1-q^n}{n(1-q)}(\bar{\theta}-\bar{b}) + \delta v \geq (1-\delta)(\bar{\theta}-\bar{b}),$$

$$\text{(HighIC-down)} \qquad (1-\delta)\frac{1-q^n}{n(1-q)}(\bar{\theta}-\bar{b}) + \delta v \geq (1-\delta)q^{n-1}(\bar{\theta}-\underline{b}),$$

$$\text{(HighIC-on-sch)} \qquad \frac{1-q^n}{n(1-q)}(\bar{\theta}-\bar{b}) \geq \frac{q^{n-1}}{n}(\bar{\theta}-\underline{b});$$

Collusiveness constraints:

$$\text{(Col-sep-1)} \qquad v \geq \frac{(1-\delta)(1-q^n)(\bar{\theta}-\underline{b})}{n(1-\delta(1-q)^n)},$$

$$\text{(Col-sep-2)} \qquad v \geq \frac{(1-\delta)\big[(1-q^n)(\bar{\theta}-\underline{b}) + q^n(\underline{\theta}-\underline{b})\big]}{n(1-\delta q^n)},$$

$$\text{(Col-pool)} \qquad v \geq \frac{1}{n}\big[(1-q)(\bar{\theta}-\underline{b}) + q(\underline{\theta}-\underline{b})\big].$$

I can now formulate the main result of the paper:

Theorem 1. *Suppose $\left(\bar{b}^*, \underline{b}^*, v^*_{\text{fse}}\right)$ solves the revenue maximization problem $\mathcal{RM}$, then there exists a critical buyers' discount factor δ^*, such that for all $\delta \in (\delta^*, 1)$ and for all $\delta_0 \in (0,1)$ the high-revenue strategy profiles corresponding to $(\bar{b}^*, \underline{b}^*)$ are collusive public perfect equilibria of the repeated auction game in all three cases (i), (ii), and (iii) of Definition 8. Moreover, $\lim_{\delta\to 1}\mathcal{R}^*_{\text{fse}}(\delta) = (1-q^n)\bar{\theta} + q^n\underline{\theta}$, i.e. the seller achieves full surplus extraction in the limit as the buyers' discount factor δ goes to 1.*

The rest of the paper will be mostly devoted to proving Theorem 1. I will divide the proof of Theorem 1 into two propositions. I will first prove the following proposition:

Proposition 1. *Suppose $\left(\bar{b}^*, \underline{b}^*, v^*_{\text{fse}}\right)$ solves the revenue maximization problem $\mathcal{RM}$. Suppose further that $\underline{\theta} < \underline{b}^* < \bar{b}^*$ and $\mathcal{R}^*_{\text{fse}}(\delta) \geq (1-q^n)\bar{\theta}$, then the high-revenue strategy profiles corresponding to $\left(\bar{b}^*, \underline{b}^*\right)$ are collusive public perfect equilibria of the repeated auction game for any seller's discount factor $\delta_0 \in (0,1)$*

(i) *in Case (i) of Definition 8,*

(ii) *if* $\delta > n\overline{\theta}/\big[n\overline{\theta} + (1-q^n - n(1-q)q^{n-1})(\overline{\theta}-\underline{\theta}) + q^{n-1}\underline{\theta}\big]$ *in Case (ii) of Definition 8,*

(iii) *if* $\delta > n\overline{\theta}/\big[n\overline{\theta} + q\underline{\theta} + (1-q)\overline{\theta} - n(1-q)q^{n-1}(\overline{\theta}-\underline{\theta})\big]$ *in Case (iii) of Definition 8.*

To prove Proposition 1, I will establish that the high-revenue strategy profiles corresponding to $(\overline{b}^*, \underline{b}^*)$ satisfy on-path incentive compatibility (Lemma 1 in Subsection 5.1), off-path incentive compatibility (Lemma 2 in Subsection 5.2), collusiveness on path (Lemma 3 in Subsection 6.1), and collusiveness off path (Lemma 5 in Subsection 6.2) in all three cases (i), (ii), and (iii) of Definition 8.

I will then prove the following proposition:

Proposition 2. *If* $\left(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}\right)$ *solves the revenue maximization problem* $\mathcal{RM}$*, then there exists a critical buyers' discount factor* δ^* *such that for all* $\delta \in (\delta^*, 1)$ *we have* $\underline{\theta} < \underline{b}^* < \overline{b}^*$*. Moreover,* $\lim_{\delta\to 1} \mathcal{R}^*_{\text{fse}}(\delta) = (1-q^n)\overline{\theta} + q^n\underline{\theta}$.

In Section 7, I will prove Proposition 2 by solving $\mathcal{RM}$, verifying that its solutions satisfy the conditions of Proposition 1 for sufficiently high δ's, and showing that the seller's revenue goes to full surplus as δ goes to 1. Section 7 will thus complete the proof of Theorem 1.

5 Incentive compatibility

5.1 On-path incentive compatibility

Lemma 1. On-path incentive compatibility

If the conditions of Proposition 1 are satisfied, then the high-revenue strategy profiles corresponding to $\left(\overline{b}^*, \underline{b}^*\right)$ *are on-path incentive compatible in all three cases (i), (ii), (iii) of Definition 8.*

Proof. Incentive compatibility of the buyers. Let us start by showing that the buyers' on-path behavior satisfies incentive compatibility constraints. Note that the following argument simultaneously covers all three cases (i), (ii), and (iii) of Definition 8. We consider two classes of deviations: *off-schedule deviations*, which involve buyers' choosing an off-path action and *on-schedule deviations*, which involve buyers' mimicking the behavior of another type.

Let us start with on-schedule deviations. The on-schedule deviation is unprofitable for a low-type buyer as long as:

$$\underbrace{\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}^*)}_{\text{On-path reward}} \geq \underbrace{\frac{1-q^n}{n(1-q)}(\underline{\theta} - \overline{b}^*)}_{\text{Mimic a high type}},$$

which is satisfied since $\underline{\theta} < \underline{b}^* < \overline{b}^*$: if a low-type buyer deviates to $\overline{b}^*$, then he receives a lower payoff with a higher probability, which cannot be profitable. The on-schedule deviation is unprofitable for a high-type buyer as long as:

$$\underbrace{\frac{1-q^n}{n(1-q)}(\overline{\theta} - \overline{b}^*)}_{\text{On-path reward}} \geq \underbrace{\frac{q^{n-1}}{n}(\overline{\theta} - \underline{b}^*)}_{\text{Mimic a low type}},$$

which is the incentive constraint (HighIC-on-sch) of the revenue maximization problem $\mathcal{RM}$ evaluated at $(\overline{b}^*, \underline{b}^*)$, and is therefore satisfied.

Consider now off-schedule deviations. First of all, we must make sure that a low-type buyer is actually willing to participate in the auction as opposed to abstaining and getting zero continuation value, i.e. that

$$\underbrace{(1-\delta)\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}^*) + \delta v^*_{\text{fse}}}_{\text{On-path payoff of a low-type buyer}} \geq (1-\delta) \underbrace{0}_{\text{Abstain}} + \delta \underbrace{0}_{\text{Switch to }\textbf{(HRP)}} = 0,$$

which is the incentive constraint (LowIC) of the revenue maximization problem $\mathcal{RM}$ evaluated at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$ and is therefore satisfied. If a low-type buyer deviates to a higher off-schedule bid, then he receives a negative expected reward in the period of the attempted deviation (since $\underline{\theta} < \underline{b}^*$) and zero continuation value, which cannot be profitable for someone who receives a positive payoff along the equilibrium path. The remaining off-schedule incentive constraints of a low-type buyer are therefore satisfied at $\left(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}\right)$.

Consider now off-schedule deviations of a high-type buyer. A high-type buyer could deviate upwards which would guarantee him winning the auction with probability 1. The best upward deviation is to $\overline{b}^* + \epsilon$, which gives the deviator a payoff almost equal to $\overline{\theta} - \overline{b}^*$. For this deviation to be unprofitable, we must have:

$$\underbrace{(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta} - \overline{b}^*) + \delta v^*_{\text{fse}}}_{\text{On-path payoff of a high-type buyer}} \geq (1-\delta) \underbrace{(\overline{\theta} - \overline{b}^*)}_{\text{Deviate to }\overline{b}^*+\epsilon} + \delta \underbrace{0}_{\text{Switch to }\textbf{(HRP)}} = (1-\delta)(\overline{\theta} - \overline{b}^*),$$

which is the incentive constraint (HighIC-up) of the revenue maximization problem $\mathcal{RM}$ evaluated at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$, and is therefore satisfied.

A high-type buyer could also deviate downwards and win the auction only in the case when all his competitors are low-type buyers, i.e. with probability q^{n-1}, or abstain from the auction altogether. Observe that $\underline{\theta} < \underline{b}^* < \overline{b}^*$ together with (LowIC) implies that $v^*_{\text{fse}} > 0$ and therefore $\overline{\theta} > \overline{b}^* > \underline{b}^*$. The best downward deviation is then to $\underline{b}^* + \epsilon$ with a payoff almost equal to $\overline{\theta} - \underline{b}^*$. For this deviation to be unprofitable, we must have:

$$\underbrace{(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*) + \delta v^*_{\text{fse}}}_{\text{On-path payoff of a high-type buyer}} \geq (1-\delta)\underbrace{q^{n-1}(\overline{\theta}-\underline{b}^*)}_{\text{Deviate to } \underline{b}^*+\epsilon} + \delta \underbrace{0}_{\text{Switch to } \textbf{(HRP)}} = (1-\delta)q^{n-1}(\overline{\theta}-\underline{b}^*),$$

which is the incentive constraint (HighIC-down) of the revenue maximization problem $\mathcal{RM}$ evaluated at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$, and is therefore satisfied.

Incentive compatibility of the seller. Consider now the seller's actions. Recall that we have $\mathcal{R}^*_{\text{fse}}(\delta) \geq (1-q^n)\overline{\theta}$ by assumption. Consider Case (i) first. In this case, a deviation to $r' \neq \underline{b}^*$ would bring the seller the revenue of $(1-\delta_0)\,\mathcal{R}^*_{\text{os},r'} + \delta_0(1-q^n)\overline{\theta}$ where $\mathcal{R}^*_{\text{os},r'}$ is the revenue achieved by her in the one-shot auction game with the reserve price equal to r'. Recall from Proposistion 0 that the optimal reserve price in the High-reserve-price region is $r_{\text{os}} = \overline{\theta}$ with the associated revenue of $(1-q^n)\overline{\theta}$. Thus the seller would not be able to get more than $(1-\delta_0)(1-q^n)\overline{\theta} + \delta_0(1-q^n)\overline{\theta} = (1-q^n)\overline{\theta} \leq \mathcal{R}^*_{\text{fse}}(\delta)$.

Consider now Cases (ii) and (iii). If the seller deviates to $r' \neq \underline{b}^*$, she will receive either $\delta_0(1-q^{n-1})\underline{\theta}$ in Case (ii), or 0 in Case (iii), neither of which can exceed $(1-q^n)\overline{\theta} \leq \mathcal{R}^*_{\text{fse}}(\delta)$ for any value of $\delta_0 \in (0,1)$. □

5.2 Off-path incentive compatibility

Lemma 2. Off-path incentive compatibility

If the conditions of Proposition 1 are satisfied, then the high-revenue strategy profiles corresponding to $(\overline{b}^*, \underline{b}^*)$ are off-path incentive compatible in all three cases (i), (ii), (iii) of Definition 8.

Proof. We consider the three cases of Definition 8 one by one.

Case (i). Observe that the play in **(One-shot)** is trivially a stronlgy symmetric public perfect equilibrium since it infinitely repeats the High-reserve-price region one-shot equilibrium. Consider the play in **(One-shot** $[r']$**)** for any r'. By Definition 8, the play switches to **(One-shot)** from **(One-shot** $[r']$**)** no matter what the buyers do in **(One-shot** $[r']$**)**. In the High-reserve-price region, the buyers' payoff in **(One-shot)** is zero, hence the buyers should play as if the game ends in **(One-shot** $[r']$**)**, i.e. play the equilibrium of the one-shot first price auction with the reserve price set to r'.

Case (ii). Observe again that the play in **(One-shot)** is trivially a stronlgy symmetric public perfect equilibrium. Consider now the play in the remaining low-revenue separating **(LRS)**, abstention **(Abstain)**, and high-reserve-price **(HRP)** states.

Incentive compatibility of the buyers. Observe first that the buyers' play in **(HRP)** is trivially incentive compatible: since the seller sets $r_{\text{hrp}} = \overline{\theta}$ regardless of what the buyers do, there cannot be a profitable deviation for the buyers in **(HRP)** (deviating downwards is impossible, deviating upwards can only lead to negative payoffs). Consider therefore the incentives of the buyers in the low-revenue-separating **(LRS)** state. Let us start with on-schedule incentive compatibility in **(LRS)**. In **(LRS)**, a low-type buyer obtains a reward of $\frac{q^{n-1}}{n}\underline{\theta}$. If a low-type buyer attempted to mimic a high-type buyer, then he would get

$$\frac{1-q^n}{n(1-q)}(\underline{\theta} - \overline{b}_{\text{lrs}}) = \frac{1-q^n}{n(1-q)}\left(\underline{\theta} - \frac{1-q^{n-1}}{1-q^n}\underline{\theta}\right) = \frac{q^{n-1}}{n}\underline{\theta}$$

as well, and hence would not have a strict incentive to do so. In **(LRS)**, a high-type buyer gets a reward of

$$\frac{1-q^n}{n(1-q)}(\overline{\theta} - \overline{b}_{\text{lrs}}) = \frac{1-q^n}{n(1-q)}\left(\overline{\theta} - \frac{1-q^{n-1}}{1-q^n}\underline{\theta}\right) = \frac{1-q^{n-1}}{n(1-q)}(\overline{\theta} - \underline{\theta}) + \frac{q^{n-1}}{n}\overline{\theta} > \frac{q^{n-1}}{n}\overline{\theta},$$

which is what he would get by mimicking a low-type buyer.

Consider now the off-schedule incentive compatibility conditions in **(LRS)**. The *ex ante* payoff of each buyer in **(LRS)** is given by:

$$v_{\text{lrs}} = \frac{1}{n}\left[(1-q^n)\left(\overline{\theta} - \frac{1-q^{n-1}}{1-q^n}\underline{\theta}\right) + q^n\underline{\theta}\right] = \frac{1}{n}\left[(1-q^n)(\overline{\theta} - \underline{\theta}) + q^{n-1}\underline{\theta}\right]$$

Recall that in the one-shot equilibrium of the Low-reserve-price region, the *ex ante* equilibrium payoff for each bidder is given by $v^*_{\text{os}} = (1-q)q^{n-1}(\overline{\theta} - \underline{\theta})$[5].

[5]Note that v_{lrs} exceeds v^*_{os} for all parameter values. Indeed, $v_{\text{lrs}} - v^*_{\text{os}} = \frac{1}{n}\left[(1-q^n)(\overline{\theta} - \underline{\theta}) + q^{n-1}\underline{\theta}\right] - (1-q)q^{n-1}(\overline{\theta} - \underline{\theta}) = \frac{1}{n}\left[(1-q^n - n(1-q)q^{n-1})(\overline{\theta} - \underline{\theta}) + q^{n-1}\underline{\theta}\right] > \frac{1}{n}q^{n-1}\underline{\theta} > 0$.

The off-schedule incentive compatibility conditions will require that the following hold in **(LRS)** for a high-type buyer:

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}_{\text{lrs}})+\delta v_{\text{lrs}} \geq (1-\delta)\max\{q^{n-1}\overline{\theta},\overline{\theta}-\overline{b}_{\text{lrs}}\}+\delta v^*_{\text{os}}.$$

The off-schedule incentive compatibility conditions will also require that the following hold in **(LRS)** for a low-type buyer:

$$(1-\delta)\frac{q^{n-1}}{n}\underline{\theta}+\delta v_{\text{lrs}} \geq (1-\delta)\max\{q^{n-1}\underline{\theta},\underline{\theta}-\overline{b}_{\text{lrs}}\}+\delta v^*_{\text{os}}.$$

Before dealing with the above two conditions, consider the incentive compatibility conditions of the buyers in **(Abstain)**. The incentive compatibility condition of a high-type buyer in **(Abstain)** is given by $\delta v_{\text{lrs}} \geq (1-\delta)(\overline{\theta}-r')+\delta v^*_{\text{os}}$. The incentive compatibility condition of a low-type buyer in **(Abstain)** is given by $\delta v_{\text{lrs}} \geq (1-\delta)(\underline{\theta}-r')+\delta v^*_{\text{os}}$. The best deviation in **(Abstain)** obtains for a high-type buyer when $r' \approx 0$. This deviation is unprofitable whenever

$$\delta\underbrace{\frac{1}{n}\big[(1-q^n)(\overline{\theta}-\underline{\theta})+q^{n-1}\underline{\theta}\big]}_{=v_{\text{lrs}}} \geq (1-\delta)\overline{\theta}+\delta\underbrace{(1-q)q^{n-1}(\overline{\theta}-\underline{\theta})}_{=v^*_{\text{os}}}. \tag{1}$$

Notice that the incentive compatibility condition in (1) implies all of the *off-schedule* incentive compatibility conditions in both **(Abstain)** and **(LRS)**. The incentive compatibility condition in (1) is satisfied for all the values of the buyers' discount factor δ such that:

$$\delta > \frac{n\overline{\theta}}{n\overline{\theta}+\big(1-q^n-n(1-q)q^{n-1}\big)(\overline{\theta}-\underline{\theta})+q^{n-1}\underline{\theta}} \equiv \delta^*_{\text{lrs}},$$

which is precisely the condition on δ required by Proposition 1 in Case (ii).

Incentive compatibility of the seller. Consider first the incentives of the seller in the low-revenue separating state **(LRS)**. The revenue of the seller in **(LRS)** is given by:

$$\mathcal{R}_{\text{lrs}} = (1-q^n)\frac{1-q^{n-1}}{1-q^n}\underline{\theta}+q^n 0 = (1-q^{n-1})\underline{\theta}.$$

It is clear that the seller does not want to deviate: if she attempts a deviation to $r'>0$, her revenue will become $(1-\delta_0)0+\delta_0(1-q^{n-1})\underline{\theta} = \delta_0(1-q^{n-1})\underline{\theta}$ (because all the buyers will abstain following $r'>0$), which cannot exceed $\mathcal{R}_{\text{lrs}}$ for any value of δ_0.

Consider now the incentives of the seller in the high-reserve-price state **(HRP)**. Her revenue in **(HRP)** is equal to $\mathcal{R}_{\text{hrp}} = (1-q^n)\overline{\theta}$. If she deviates to any $r' \neq \overline{\theta}$, then her revenue is $(1-\delta_0)0+\delta_0(1-q^{n-1})\underline{\theta} = \delta_0(1-q^{n-1})\underline{\theta} < (1-q^n)\overline{\theta} = \mathcal{R}_{\text{hrp}}$.

Case (iii). As in the previous cases, the play in **(One-shot)** is trivially a strongly symmetric public perfect equilibrium. It therefore remains to consider the play in the zero-revenue pooling **(ZRP)**, abstention **(Abstain)**, and high-reserve-price **(HRP)** states.

Incentive compatibility of the buyers. As in Case (ii), the buyers' play in **(HRP)** is trivially incentive compatible since the seller sets $r_{\mathrm{hrp}} = \overline{\theta}$ no matter what they do. Consider therefore the buyers' play in the zero-revenue pooling state **(ZRP)**. A buyer's *ex ante* payoff in **(ZRP)** is equal to:

$$v_{\mathrm{zrp}} = \frac{1}{n}\big[(1-q)\overline{\theta} + q\underline{\theta}\big].$$

The best deviation available at **(ZRP)** is for a high-type buyer to bid $0+\epsilon$ for some small ϵ. The associated incentive compatibility condition is:

$$(1-\delta)\frac{1}{n}\overline{\theta} + \delta v_{\mathrm{zrp}} \geq (1-\delta)\overline{\theta} + \delta v_{\mathrm{os}}^{*}. \tag{2}$$

Consider now the buyers' play in **(Abstain)**. The best deviation available at **(Abstain)** is for the a high-type buyer to bid r' and obtain the payoff of $\overline{\theta} - r'$. The associated incentive compatibility condition is given by $\delta v_{\mathrm{zrp}} \geq (1-\delta)(\overline{\theta} - r') + \delta v_{\mathrm{os}}^{*}$. Clearly this deviation is most profitable when $r' \approx 0$, therefore we could rule out all such deviations if we made sure that the following condition holds:

$$\delta \underbrace{\frac{1}{n}\big[(1-q)\overline{\theta} + q\underline{\theta}\big]}_{=v_{\mathrm{zrp}}} \geq (1-\delta)\overline{\theta} + \delta \underbrace{(1-q)q^{n-1}(\overline{\theta} - \underline{\theta})}_{=v_{\mathrm{os}}^{*}}. \tag{3}$$

Observe that the incentive compatibility condition in (3) also implies the incentive compatibility condition at **(ZRP)** derived in (2), and thus the buyers' play in Case (iii) is off-path incentive compatible as long as the condition in (3) is satisfied. It is satisfied whenever

$$\delta > \frac{n\overline{\theta}}{n\overline{\theta} + q\underline{\theta} + (1-q)\overline{\theta} - n(1-q)q^{n-1}(\overline{\theta} - \underline{\theta})} \equiv \delta_{\mathrm{zrp}}^{*}. \tag{4}$$

Note that the critical value of the buyers' discount factor $\delta_{\mathrm{zrp}}^{*}$ defined in (4) is in $(0,1)$ as long as $q\underline{\theta} + (1-q)\overline{\theta} - n(1-q)q^{n-1}(\overline{\theta} - \underline{\theta}) = n(v_{\mathrm{zrp}} - v_{\mathrm{os}}^{*})$ is strictly positive. Recall that the low-revenue-separating payoff v_{lrs} from Case (ii) always exceeds the one-shot equilibrium payoff v_{os}^{*} in the Low-reserve-price region. Recall also that in Case (iii) we have $q < \frac{\underline{\theta}}{\overline{\theta} - \underline{\theta}}$, and hence $v_{\mathrm{zrp}} > v_{\mathrm{lrs}} > v_{\mathrm{os}}^{*}$[6], which establishes that $\delta_{\mathrm{zrp}}^{*} \in (0,1)$ in Case (iii).

[6]Indeed, comparing v_{zrp} and v_{lrs}, we get $v_{\mathrm{lrs}} - v_{\mathrm{zrp}} = \frac{1}{n}\big[(1-q^{n})(\overline{\theta} - \underline{\theta}) + q^{n-1}\underline{\theta}\big] - (1-q)\frac{1}{n}\overline{\theta} - q\frac{1}{n}\underline{\theta} = \frac{1-q^{n-1}}{n}\big[q\overline{\theta} - (1+q)\underline{\theta}\big]$, which means that $v_{\mathrm{lrs}} < v_{\mathrm{zrp}}$ whenever $q < \frac{\underline{\theta}}{\overline{\theta} - \underline{\theta}}$.

Incentive compatibility of the seller. Consider first the seller's incentives in **(ZRP)**. The seller does not have any incentive to deviate because she would end up with zero regardless of the reserve price, which makes setting $r_{\text{zrp}} = 0$ one of her optimal choices.

Consider now the seller's incentives in the high-reserve-price state **(HRP)**. Just as in Case (ii), the seller's revenue is equal to $\mathcal{R}^*_{\text{hrp}} = (1 - q^n)\overline{\theta}$. She does not want to deviate since a deviation would lead to zero revenue forever. □

5.3 Summary of Section 5

Lemma 1 and Lemma 2 together imply that the players' high-revenue strategy profiles corresponding to $(\overline{b}^*, \underline{b}^*)$ are strongly symmetric public perfect equilibria in all three cases (i), (ii), and (iii) of Definition 8. In the remainder of the paper, we will refer to them as *high-revenue equilbiria corresponding to* $(\overline{b}^*, \underline{b}^*)$. Lemma 1 and Lemma 2 also in particular imply that the players' continuation strategy profiles in the **(LRS)**, **(ZRP)**, and **(HRP)** states are strongly symmetric public perfect equilibria as well. In the remainder of the paper, we will refer to these continuation profiles as *low-revenue separating*, *zero-revenue pooling*, and *high-reserve-price* equilibria respectively.

6 Collusiveness

6.1 Collusiveness on path

Lemma 3. Collusiveness on path

If the conditions of Proposition 1 are satisfied, then the high-revenue equilibria corresponding to $(\overline{b}^*, \underline{b}^*)$ *are collusive on path in all three cases (i), (ii), (iii) of Definition 8.*

Proof. Let us consider the buyer-games induced by the seller's equilibrium strategies. Note that these buyer-games are exactly the same stochastic game (up to parameter values) in all three cases (i), (ii), (iii) of Definition 8. This stochastic game has two states. It starts in the low-reserve-price state ω^l, in which the reserve price is $r(\omega^l) = \underline{b}^*$, and remains in that state unless a bid outside of $\{\underline{b}^*, \overline{b}^*\}$ is placed by at least one buyer, in which case the game transitions to the high-reserve-price state ω^h, in which the reserve price is $r(\omega^h) = \overline{\theta}$. The high-reserve-price state ω^h is absorbing, i.e. once the high reserve-price-state is achieved, the game remains in that state forever. The full formal definition is as follows:

Definition 9. High-revenue buyer-game

- *The set of actions for each buyer is A, i.e. is as defined in the repeated auction game.*
- *The set of states is $\Omega = \{\omega^l, \omega^h\}$, the initial state is $\omega^0 = \omega^l$.*
- *The transitions between states occur according to τ:*

$$\tau(\omega^l, b) = \begin{cases} \omega^l, & \text{if } b \in \{\underline{b}^*, \overline{b}^*\}^n \\ \omega^h, & \text{otherwise} \end{cases} ; \qquad \tau(\omega^h, b) = \omega^h \quad \forall b.$$

- *The set of valuations is Θ, i.e. is as defined in the repeated auction game.*
- *The utility of buyer i with type θ_i bidding b_i in state ω is*

$$\tilde{u}_i(\omega, b, \theta_i) = \begin{cases} \frac{1}{\#(\text{win})}(\theta_i - b_i), & \text{if } b_i \geq r(\omega) \text{ \& } \big(b_i = \max\{b_1, ..., b_n\} \text{ or } b_{-i} = \emptyset\big) \\ 0, & \text{otherwise} \end{cases},$$

where $\#(\text{win})$ is the number of winners in the auction, $r(\omega^l) = \underline{b}^*$, $r(\omega^h) = \overline{\theta}$.

The definition of collusiveness on path (Definition 5) requires that the buyers be unable to play a strongly symmetric public perfect equilibrium in the high-revenue buyer-game of Definition 9 that would improve their payoff. I first show that the buyers' strategy in any strongly symmetric public perfect equilibrium of this buyer-game must be monotonic:

Lemma 4. Monotonicity

Consider the high-revenue buyer-game in Definition 9. Any strongly symmetric public perfect equilibrium of this buyer-game satisfies monotonicity: pick any history of play that leads to state ω_l, if $\overline{b}$ is the equilibrium bidding action of a high-type buyer and $\underline{b}$ is the equilibrium bidding action of a low-type buyer after that history, then either $\overline{b} \geq \underline{b}$, or $\overline{b} \neq \emptyset$ and $\underline{b} = \emptyset$, or $\overline{b} = \underline{b} = \emptyset$.

Proof. See Appendix B. □

The *ex ante* payoff from bidding $(\overline{b}, \underline{b})$ in state ω^l is given by:

$$\hat{u}_{\omega^l}(\overline{b}, \underline{b}) \equiv \begin{cases} \frac{1}{n}\big[(1-q^n)(\overline{\theta} - \overline{b}) + q^n(\underline{\theta} - \underline{b})\big] & \text{if } \overline{b} > \underline{b}, \\ \frac{1}{n}\big[(1-q)(\overline{\theta} - \underline{b}) + q(\underline{\theta} - \underline{b})\big] & \text{if } \overline{b} = \underline{b}, \end{cases}$$

where whenever $\underline{b} = \emptyset$ or $\overline{b} = \emptyset$, the convention is to set the *ex post* payoff to zero.

When the current state is ω^h, the reserve price is equal to $\overline{\theta}$, and thus the buyers cannot get more than zero in any continuation equilibrium in that state. Since they cannot get a negative payoff in any continuation equilibrium either, they must be getting zero once the game is stuck in state ω^h, we therefore set $\hat{u}_{\omega^h}(\overline{b}, \underline{b}) \equiv 0$ without loss of generality.

Consider now the relaxed optimal collusion problem in the high-revenue buyer-game, which ignores all the aspects of incentive compatibility except monotonicity. It is given by:

$$\sup_{\{\overline{b}_t, \underline{b}_t\}_{t=0}^{+\infty}} (1-\delta) \sum_{t=0}^{+\infty} \delta^t \hat{u}_\omega(\overline{b}_t, \underline{b}_t) \text{ subject to}$$
$$\overline{b}_t \geq \underline{b}_t \; \forall t; \quad \text{transition function } \tau \text{ as in Definition 9.}$$

This optimization problem gives us an upper bound on strongly symmetric public perfect equilibrium payoffs in the high-revenue buyer-game of Definition 9. It follows from Blackwell (1965) that, if this problem has a solution, it must also have a stationary solution. I therefore consider two kinds of stationary monotonic bidding profiles: *separating* and *pooling*.

Separating profiles. If both types bid *on schedule*, then clearly there is only one option: $\underline{b} = \underline{b}^*$ and $\overline{b} = \overline{b}^*$ with the payoff equal to v^*_{fse}. *If all buyers of type $\overline{\theta}$ bid on schedule and all buyers of type $\underline{\theta}$ bid off schedule*, then the off-schedule action of any low-type buyer will be immediately punished with zero continuation values. Since the punishment will not occur if and only if all the buyers have high types (i.e. with probability $(1-q)^n$), the resulting *ex ante* payoff will be:

$$v(\overline{b}, \underline{b}) = (1-\delta)\frac{1}{n}\left[(1-q^n)(\overline{\theta} - \overline{b}) + q^n(\underline{\theta} - \underline{b})\right] + \delta(1-q)^n v(\overline{b}, \underline{b}).$$

Recall that we assume $\underline{b}^* > \underline{\theta}$, hence by incentive compatibility we must have $\overline{b}^* < \overline{\theta}$. Then the optimal solution here is to coordinate on the bidding profile in which any high-type buyer bids the low equilibrium bid $\underline{b}^*$ and any low-type buyer abstains, with the payoff of

$$v(\underline{b}^*, \emptyset) = \frac{(1-\delta)\left[(1-q^n)(\overline{\theta} - \underline{b}^*)\right]}{n(1-\delta(1-q)^n)}.$$

The collusiveness constraint (Col-sep-1) of the revenue maximization problem $\mathcal{RM}$ evaluated at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$ gives us $v^*_{\text{fse}} \geq v(\underline{b}^*, \emptyset)$.

If all buyers of type $\overline{\theta}$ bid off schedule and all buyers of type $\underline{\theta}$ bid on schedule, then the off-schedule action of any high-type buyer will be punished with zero continuation values.

Since the punishment will not occur if and only if all the buyers have low types (i.e. with probability q^n), the resulting payoff will be:

$$v(\bar{b}, \underline{b}) = (1-\delta)\frac{1}{n}\big[(1-q^n)(\bar{\theta}-\bar{b}) + q^n(\underline{\theta}-\underline{b})\big] + \delta q^n v(\bar{b}, \underline{b}).$$

The optimal solution here is for the low types to choose $\underline{b}^*$ and for the high types to choose $\underline{b}^* + \epsilon$, with the resulting payoff of

$$v(\underline{b}^* + \epsilon, \underline{b}^*) = \frac{(1-\delta)\big[(1-q^n)(\bar{\theta}-\underline{b}^*) + q^n(\underline{\theta}-\underline{b}^*)\big]}{n(1-\delta q^n)}.$$

The collusiveness constraint (Col-sep-2) of the revenue maximization problem $\mathcal{RM}$ evaluated at $(\bar{b}^*, \underline{b}^*, v^*_{\text{fse}})$ gives us $v^*_{\text{fse}} \geq v(\underline{b}^* + \epsilon, \underline{b}^*)$.

If buyers of both types bid off schedule, then they will be punished with zero continuation values with probability 1, and the resulting payoff will be:

$$v(\bar{b}, \underline{b}) = (1-\delta)\frac{1}{n}\big[(1-q^n)(\bar{\theta}-\bar{b}) + q^n(\underline{\theta}-\underline{b})\big] + \delta 0.$$

Since it must be that $\bar{b}^* < \bar{\theta}$, the best bidding profile here is for the high types to choose $\underline{b}^* + \epsilon$ and for the low types to abstain, which results in:

$$v(\underline{b}^* + \epsilon, \emptyset) = (1-\delta)\frac{1}{n}(1-q^n)(\bar{\theta}-\underline{b}^*),$$

which is clearly below $v(\underline{b}^*, \emptyset)$ and therefore below v^*_{fse}.

Pooling profiles. The buyers might find it optimal to pool instead of separating. *If the buyers pool on schedule*, then their collusive scheme is never detected. Clearly the optimal pooling on schedule is achieved at $\underline{b}^*$ with the resulting payoff of

$$v(\underline{b}^*, \underline{b}^*) = \frac{1}{n}\big[(1-q)(\bar{\theta}-\underline{b}^*) + q(\underline{\theta}-\underline{b}^*)\big] \tag{5}$$

The collusiveness constraint (Col-pool) of the revenue maximization problem $\mathcal{RM}$ evaluated at $(\bar{b}^*, \underline{b}^*, v^*_{\text{fse}})$ gives us $v^*_{\text{fse}} \geq v(\underline{b}^*, \underline{b}^*)$.

Note that the payoff from *pooling off-schedule* cannot exceed v^*_{fse}. If the buyers coordinate on any off-schedule bid above $\underline{b}^*$ they will get a fraction of the payoff in (5). Abstaining altogether cannot be optimal as long as $v^*_{\text{fse}} > 0$, which it is by incentive compatibility.

We therefore conclude that no strongly symmetric public perfect equilibrium payoff in the high-revenue buyer-game corresponding to $(\bar{b}^*, \underline{b}^*)$ can exceed v^*_{fse}, and therefore the high-revenue equilibrium corresponding to $(\bar{b}^*, \underline{b}^*)$ is collusive on path. $\square$

6.2 Collusiveness off path

Lemma 5. Collusiveness off path

If the conditions of Proposition 1 are satisfied, then the high-revenue equilibria corresponding to $\left(\overline{b}^*, \underline{b}^*\right)$ *are collusive off path in all three cases (i), (ii), (iii) of Definition 8.*

Proof. We consider the three cases of Definition 8.

Case (i). According to the definition of collusiveness off path (Definition 6), we need to check that the inifinite repetition of the one-shot equilibrium of the High-reserve-price region is collusive *on path*. Recall from Remark 1 that this is indeed the case for any values of the discount factors δ_0 and δ.

Cases (ii) and (iii). According to the definition of collusiveness off path (Definition 6), we need to check that, the low-revenue separating equilibrium, the zero-revenue pooling equilibrium, and the high-reserve-price equilibria of Cases (ii) and (iii) are collusive *on path*. Observe first that the argument of Remark 1 immediately extends to the two high-reserve-price equilibria, so they are clearly collusive *on path*. Observe also that both the low-revenue separating equilibrium of Case (ii) and the zero-revenue pooling equilibrium of Case (iii) lead to the same buyer-game. This buyer game is a repeated first-price auction game, in which the reserve price is set to zero. The following is its formal definition:

Definition 10. Low-revenue buyer-game

- *The set of actions for each buyer is* A*, i.e. as defined in the repeated auction game.*
- *The set of states is* $\Omega = \{\omega^0\}$*, the transition function is trivial:* $\tau(\omega^0, b) = \omega^0$ *for all* b*.*
- *The set of valuations is* Θ*, i.e. is as defined in the repeated auction game.*
- *The utility of buyer* i *with type* θ_i *bidding* b_i *in state* ω *is*

$$\tilde{u}_i(\omega, b, \theta_i) = \begin{cases} \frac{1}{\#(\text{win})}(\theta_i - b_i), & \text{if } b_i = \max\{b_1, ..., b_n\} \text{ or } b_{-i} = \emptyset \\ 0, & \text{otherwise} \end{cases},$$

where $\#(\text{win})$ *is the number of winners in the auction.*

The following lemma applies:

Lemma 6. On-path collusiveness of low-revenue equilibria

Consider the low-revenue buyer-game in Definition 10. In Case (ii), the maximal strongly symmetric public perfect equilibrium payoff achievable by the buyers is equal to their low-revenue separating equilbrium payoff. In Case (iii), the maximal strongly symmetric public perfect equilibrium payoff achievable by the buyers is equal to their zero-revenue pooling payoff. Hence the low-revenue separating and the zero-revenue pooling equilbria are collusive on path in their respective cases.

The proof of Lemma 6 is relegated to Appendix C. The main idea is to show (by relaxing some of the incentive compatibility constraints) that if the buyers find it optimal to separate (or, respectively, pool) in the first period, then they will find it optimal to separate (or, respectively, pool) in every period along the equilibrium path. The optimal pooling is then trivially at zero, and the optimal separation is pinned down by the *on-schedule* incentive compatibility constraint of a low-type buyer. □

7 Revenue maximization and full surplus extraction

Let us now turn our attention to Proposition 2. To prove Proposition 2, we are going to solve the revenue maximization problem $\mathcal{RM}$ and verify that its solutions indeed define *collusive public perfect equilibria* that allow the seller to extract full surplus from the buyers in the limit. To solve $\mathcal{RM}$, we will distinguish three more cases depending on which of the $\mathcal{RM}$'s constraints are binding at the optimum. To avoid confusion with the three cases of Definition 8, we will use Arabic numerals to label the solutions of $\mathcal{RM}$. The parameter regions corresponding to each case are illustrated by Figure 3. Appendix H contains additional results characterizing the three parameter regions.

In Case 1, the (Col-sep-1) and (LowIC) constraints are binding. Case 1 does not always apply because its solution candidate does not always satisfy the (HighIC-up) incentive compatibility constraint: if n is high enough, the winning probability of a high-type buyer is so low that such a buyer would prefer to win with probability 1 by placing a slightly higher bid and suffer the punishment of zero continuation value. We therefore have to consider Case 2, in which (HighIC-up) and (Col-sep-1) are binding and the remaining constraints are slack. Case 2's equilibrium candidate in turn does not apply for high values of q: in this case the

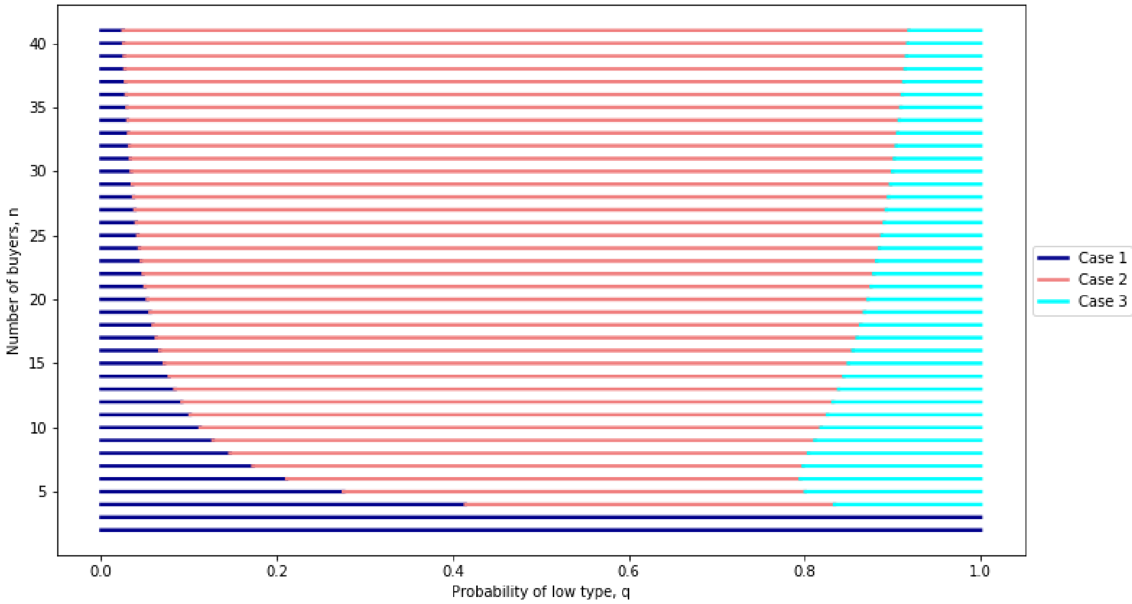


Figure 3: *Parameter regions corresponding to Cases 1, 2, and 3. For each number of buyers n, the respective line shows which values of q belong to Cases 1, 2, and 3.*

(HighIC-down) incentive compatibility constraint will be violated. Intuitively, if the mass of low types is sufficiently large, then a high-type buyer will have a fairly high chance of winning by bidding just above the low-type equilibrium bid even though placing such a bid is severely punished. In Case 3, the (HighIC-up) and (HighIC-down) incentive compatibility constraints are binding.

The remaining constraints in the revenue maximization problem are non-binding in all three cases. To develop some intuition, consider first the on-schedule incentive compatibility constraint of a high-type buyer (HighIC-on-sch). This constraint essentially puts an upper bound on the high-type equilibrium bid (if a high-type buyer is asked to bid a lot more than a low-type buyer, he might find it profitable to deviate to the low-type bid and get a much higher reward with a smaller winning probability), but we have already included a constraint that does the same, the collusiveness constraint (Col-sep-1). Indeed, if a high-type buyer is asked to place a very high bid in every period, then the buyers might find it profitable to collude on a lower bidding profile, and such a collusion scheme is prevented by (Col-sep-1). The restriction on equilibrium bids imposed by (Col-sep-1) is more severe than the one imposed by the on-schedule incentive compatibility of a high type buyer. Clearly, if the more severe restriction had been the one imposed by incentive compatibility, we would have been unlikely to consider collusion an important problem in the first place.

The two remaining collusiveness constraints, (Col-sep-2) and (Col-pool), are also non-binding in all three cases, which means that the optimal collusion scheme for the buyers always involves bidding $\underline{b}^*$ for the high types and abstaining for the low types. Collusion by pooling on schedule turns out to be particularly inefficient as it leads to negative payoffs for the buyers for δ's close to 1. Collusion by leaving the low types on schedule and moving the high types off schedule does not outperform the optimal collusion scheme because it leads to punishments for the high types, who, unlike the low types, get a positive payoff in every period, thus the gain from bidding lower made by the high types is completely offset by the severity of the punishment.

In the remainder of this section, I will solve the revenue maximization problem $\mathcal{RM}$ using a relaxed-problem approach. I will construct three relaxed revenue maximization problems and show that each of them has a solution that satisfies $\underline{\theta} < \underline{b}^* < \overline{b}^*$ for sufficiently high values of δ. I will then show that each of the three solutions is also a solution to $\mathcal{RM}$ by checking the remaining constraints. In all three cases the seller will be able to extract full surplus from the buyers in the limit as the buyers' discount factor δ goes to 1.

Case 1: High expected valuation/Small number of buyers

$$q < \frac{1-q^n}{n(1-q)}$$

In Case 1, we consider the following relaxed maximization problem:

$$\mathcal{RM}\text{-1}: \quad \max_{\overline{b},\underline{b},v}(1-q^n)\overline{b} + q^n\underline{b}, \quad \text{s.t.}$$

$$\text{(Eq-payoff)}, \ \text{(LowIC)}, \ \text{(Col-sep-1)};$$

and first establish the following lemma:

Lemma 7. *There exists a critical discount factor δ^* such that for all $\delta \in (\delta^*, 1)$ $\mathcal{RM}$-1 has* *an optimum, at which* (LowIC) *and* (Col-sep-1) *are binding. The optimum is*[7]*:*

$$\underline{b}^* = \underline{\theta} + \frac{\delta q\big(1-q^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}, \tag{6}$$

$$\overline{b}^* = \overline{\theta} - \frac{q^n(1-\delta(1-q))(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}, \tag{7}$$

$$v^*_{\text{fse}} = \frac{1}{n}\frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}. \tag{8}$$

Proof. See Appendix D.1. □

We then show that the bids in (6) and (7) satisfy the conditions of Proposition 1:

Lemma 8. $\underline{\theta} < \underline{b}^* < \overline{b}^*$*.*

Proof. $\underline{\theta} < \underline{b}^*$ is equivalent to $\underline{\theta} - \underline{b}^* < 0$, which is true since $-\delta q(1-q^n)(\overline{\theta}-\underline{\theta}) < 0$. $\underline{b}^* < \overline{b}^*$ is equivalent to $\overline{\theta} - \underline{b}^* > \overline{\theta} - \overline{b}^*$. Observe that

$$\overline{\theta} - \underline{b}^* = \frac{q^n\big(1-\delta(1-q)^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)},$$

hence $\overline{\theta} - \underline{b}^* > \overline{\theta} - \overline{b}^*$ is equivalent to $1-\delta(1-q)^n > 1-\delta(1-q)$, which is true for any $\delta \in (0,1)$ and $q \in (0,1)$ since $n \geq 2$. □

And finally, we show that the solution to $\mathcal{RM}$-1 is also a solution to the revenue maximization problem $\mathcal{RM}$ for sufficiently high values of the buyers' discount factor:

Lemma 9. *Suppose that* $q < \frac{1-q^n}{n(1-q)}$*. Suppose further that* $\underline{b}^*$*,* $\overline{b}^*$*, and* v^*_{fse} *are as defined in* (6)*,* (7)*, and* (8) *respectively, then there exists a critical buyers' discount factor* δ^**, such that for all* $\delta \in (\delta^*, 1)$ *and for all* $\delta_0 \in (0,1)$ *the tuple* $\big(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}\big)$ *solves* $\mathcal{RM}$*, implying in turn that the high-revenue strategy profile corresponding to* $(\overline{b}^*, \underline{b}^*)$ *is a collusive public perfect equilibrium of the repeated auction game. Moreover, the seller achieves full surplus extraction in the limit as* δ *goes to* 1*.*

Proof sketch. The complete proof is provided in Appendix G.1. Here I briefly sketch the most important points. To show full surplus extraction, recall that the seller's revenue is equal to

[7] The equilibrium conditions are thus given by a system of linear equations, whose solution is presented in Appendix E.1.

the full surplus net of the equilibrium payoff of the buyers: $\mathcal{R}^*_{\text{fse}}(\delta) = (1-q^n)\overline{\theta} + q^n\underline{\theta} - nv^*_{\text{fse}}$. It is easy to see from (8) that $nv^*_{\text{fse}} \xrightarrow[\delta \to 1]{} 0$.

To show that the high-revenue strategy profile corresponding to $\left(\overline{b}^*, \underline{b}^*\right)$ is a collusive public perfect equilibrium, recall that by Proposition 1 and Lemma 8, it is enough to check that $\mathcal{R}^*_{\text{fse}}(\delta) \geq (1-q^n)\overline{\theta}$, and that the remaining constraints in the revenue maximization problem $\mathcal{RM}$ are satisfied at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$ for high enough δ. $\mathcal{R}^*_{\text{fse}}(\delta)$ clearly exceeds $(1-q^n)\overline{\theta}$ for high enough values of δ due to full surplus extraction in the limit, and the remaining constraints are checked by direct calculation.

All of the remaining constraints in $\mathcal{RM}$ are non-binding at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$ for all δ high enough and all values of q and n, except the incentive constraint (HighIC-up). There is a region of q and n, where this constraint cannot be satisfied even for δ close to 1. To see why, observe that (HighIC-up) can be rewritten as:

$$\delta v^*_{\text{fse}} \geq (1-\delta)\left(1 - \frac{1-q^n}{n(1-q)}\right)(\overline{\theta} - \overline{b}^*)$$

Plugging the respective expressions from (7) and (8) into the above inequality, we obtain:

$$\delta \geq \frac{1}{1-q} - \frac{1-q^n}{n(1-q)^2}. \tag{9}$$

The condition on δ identified in (9) can only be satisfied if the right-hand side of (9) is strictly below 1, which is only true when $q < \frac{1-q^n}{n(1-q)}$, hence the parameter region of Case 1.

For the rest of the constraints, see Appendix G.1. □

Case 2: Medium expected valuation[8]

$$\frac{1-q^n}{n(1-q)} \leq q < 1 - \frac{q^{n-1}\left(1-(1-q)^n\right)\left[n(1-q)-(1-q^n)\right]}{1-q^n}$$

In Case 2, we consider the following relaxed maximization problem:

$$\mathcal{RM}\text{-}2: \quad \max_{\overline{b},\underline{b},v}(1-q^n)\overline{b} + q^n\underline{b}, \quad \text{s.t.}$$

$$\text{(Eq-payoff)}, \text{(HighIC-up)}, \text{(Col-sep-1)};$$

and first establish the following lemma:

[8]To see that the parameter region is well-defined, notice that $nq^{n-1} < \sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q}$ implies that $q^{n-1}\left(1-(1-q)^n\right) < \frac{1-q^n}{n(1-q)}$, which is in turn equivalent to $\frac{1-q^n}{n(1-q)} < 1 - \frac{q^{n-1}\left(1-(1-q)^n\right)\left[n(1-q)-(1-q^n)\right]}{1-q^n}$.

Lemma 10. *There exists a critical discount factor δ^* such that for all $\delta \in (\delta^*, 1)$ $\mathcal{RM}$-2 has an optimum, at which (HighIC-up) and (Col-sep-1) are binding. The optimum is*[9]*:*

$$\underline{b}^* = \underline{\theta} + \frac{1}{D(\delta)}\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big](1-q^n)(\overline{\theta} - \underline{\theta}), \tag{10}$$

$$\overline{b}^* = \overline{\theta} - \frac{1}{D(\delta)}\delta q^n (1-q^n)(1-q)(\overline{\theta} - \underline{\theta}), \tag{11}$$

$$v^*_{\text{fse}} = \frac{1-\delta}{nD(\delta)} q^n (1-q^n)\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}), \tag{12}$$

where

$$D(\delta) \equiv q^n\big(1-\delta(1-q)^n\big)\big[n(1-q) - (1-q^n)\big] + (1-q^n)\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big].$$

Proof. See Appendix D.2. □

We then show that the bids in (10) and (11) satisfy the conditions of Proposition 1 for sufficiently high δ:

Lemma 11. *Suppose $q \geq \frac{1-q^n}{n(1-q)}$, and δ is sufficiently close to 1, then $\underline{\theta} < \underline{b}^* < \overline{b}^*$.*

Proof. To see that $\underline{\theta} < \underline{b}^*$ for sufficiently high δ, observe that

$$\underline{\theta} - \underline{b}^* \xrightarrow[\delta \to 1]{} -\frac{1}{D(1)}(1-q^n)(1-q)(1-q^n)(\overline{\theta} - \underline{\theta}) < 0.$$

The proof of $\underline{b}^* < \overline{b}^*$ is provided in Appendix F. □

And finally, we show that the solution to $\mathcal{RM}$-2 is also a solution to the revenue maximization problem $\mathcal{RM}$ for sufficiently high values of the buyers' discount factor:

Lemma 12. *Suppose that $\frac{1-q^n}{n(1-q)} \leq q < 1 - \frac{q^{n-1}(1-(1-q)^n)[n(1-q)-(1-q^n)]}{1-q^n}$. Suppose further that $\underline{b}^*$, $\overline{b}^*$, and v^*_{fse} are as defined in (10), (11), and (12) respectively, then there exists a critical buyers' discount factor δ^*, such that for all $\delta \in (\delta^*, 1)$ and for all $\delta_0 \in (0, 1)$ the tuple $\big(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}\big)$ solves $\mathcal{RM}$, implying in turn that the high-revenue strategy profile corresponding to $(\overline{b}^*, \underline{b}^*)$ is a collusive public perfect equilibrium of the repeated auction game. Moreover, the seller achieves full surplus extraction in the limit as δ goes to 1.*

[9] The equilibrium conditions are thus given by a system of linear equations, whose solution is presented in Appendix E.2.

Proof sketch. The complete proof is provided in Appendix G.2. As in the previous case, I only provide a sketch of the most important points in the main text. The argument for full surplus extraction is exactly the same as in Lemma 9: from (12) we have $nv^*_{\text{fse}} \xrightarrow[\delta\to 1]{} 0$, implying $\mathcal{R}^*_{\text{fse}}(\delta) \xrightarrow[\delta\to 1]{} (1-q^n)\overline{\theta} + q^n\underline{\theta}$. As in the previous case, full surplus extraction in the limit implies that $\mathcal{R}^*_{\text{fse}}(\delta) \geq (1-q^n)\overline{\theta}$ for all high enough values of δ, hence Proposition 1 and Lemma 11 imply that the high-revenue strategy profile corresponding to $\left(\overline{b}^*, \underline{b}^*\right)$ is a collusive public perfect equilibrium as long as the remaining constraints in the revenue maximization problem $\mathcal{RM}$ are satisfied at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$ for high enough δ. The remaining constraints are checked by direct calculation.

Let us first check that (LowIC) is satisfied under the parameter restriction of Case 2. Recall that a low-type buyer must be willing to participate in the bidding with the bid $\underline{b}^*$ as opposed to abstaining and getting a zero payoff:

$$(1-\delta)\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}^*) + \delta v^*_{\text{fse}} \geq 0.$$

Plugging the respective expressions from (10) and (12) into the above inequality, we obtain:

$$\delta \leq \frac{1}{1-q} - \frac{1-q^n}{n(1-q)^2},$$

which is true since $\frac{1}{1-q} - \frac{1-q^n}{n(1-q)^2} \geq 1$ by assumption that $q \geq \frac{1-q^n}{n(1-q)}$.

The remaining constraints of $\mathcal{RM}$ are all non-binding at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$ for high values of δ and all values of q and n, except for the constraint associated with a downward deviation of a high-type buyer (HighIC-down). Recall that a high-type buyer could deviate to $\underline{b}^* + \epsilon$ and win whenever all of his competitors are low types. For this deviation to be unprofitable, his payoff must satisfy:

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta} - \overline{b}^*) + \delta v^*_{\text{fse}} \geq (1-\delta)q^{n-1}(\overline{\theta} - \underline{b}^*).$$

Plugging expressions (10), (11), and (12) into the above inequality, we obtain:

$$\delta(1-q^n)(1-q) \geq q^{n-1}\left(1-\delta(1-q)^n\right)\left[n(1-q) - (1-q^n)\right],$$

which is satisfied for δ high enough as long as:

$$(1-q^n)(1-q) > q^{n-1}\left(1-(1-q)^n\right)\left[n(1-q) - (1-q^n)\right],$$

which is true since $q < 1 - \frac{q^{n-1}(1-(1-q)^n)[n(1-q)-(1-q^n)]}{1-q^n}$ in Case 2.

The rest of the constraints are checked in Appendix G.2. ☐

Case 3: Low expected valuation

$$q \geq 1 - \frac{q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]}{1-q^n}$$

In Case 3 we consider the following relaxed maximization problem:

$$\mathcal{RM}\text{-}3: \quad \max_{\bar{b},\underline{b},v}(1-q^n)\bar{b} + q^n\underline{b}, \quad \text{s.t.}$$

$$\text{(Eq-payoff)}, \ \text{(HighIC-up)}, \ \text{(HighIC-down)};$$

and first establish the following lemma:

Lemma 13. *There exists a critical discount factor δ^* such that for all $\delta \in (\delta^*, 1)$ $\mathcal{RM}$-3 has an optimum, at which (HighIC-up) and (HighIC-down) are binding. The optimum is*[10]*:*

$$\underline{b}^* = \underline{\theta} + \frac{1}{D(\delta)}\big[(1-q^n)(1-\delta q) - n(1-\delta)(1-q)\big](\bar{\theta}-\underline{\theta}), \tag{13}$$

$$\bar{b}^* = \bar{\theta} - \frac{1}{D(\delta)}\delta q^n(1-q)(\bar{\theta}-\underline{\theta}), \tag{14}$$

$$v^*_{\text{fse}} = \frac{1}{nD(\delta)}(1-\delta)q^n\big[n(1-q)-(1-q^n)\big](\bar{\theta}-\underline{\theta}), \tag{15}$$

where

$$D(\delta) \equiv (1-q^n)(1-\delta q) + \delta q(1-q) - n(1-\delta)(1-q).$$

Proof. See Appendix D.3. □

As in the previous two cases, we then show that the bids in (13) and (14) satisfy the conditions of Proposition 1 for sufficiently high δ:

Lemma 14. *Suppose δ is sufficiently close to 1, then $\underline{\theta} < \underline{b}^* < \bar{b}^*$.*

Proof. To see that $\underline{\theta} < \underline{b}^*$ for sufficiently high values of δ, observe that:

$$\underline{\theta} - \underline{b}^* \xrightarrow[\delta\to 1]{} -\frac{1}{D(1)}(1-q^n)(1-q)(\bar{\theta}-\underline{\theta}) < 0.$$

$\underline{b}^* < \bar{b}^*$ is equivalent to $\bar{\theta} - \underline{b}^* > \bar{\theta} - \bar{b}^*$. Observe that $\bar{\theta} - \underline{b}^* = \frac{1}{D(\delta)}\delta q(1-q)(\bar{\theta}-\underline{\theta})$, hence $\bar{\theta} - \underline{b}^* > \bar{\theta} - \bar{b}^*$ is equivalent to:

$$\frac{1}{D(\delta)}\delta q(1-q)(\bar{\theta}-\underline{\theta}) > \frac{1}{D(\delta)}\delta q^n(1-q)(\bar{\theta}-\underline{\theta}),$$

which is clearly true since $D(\delta) > 0$ for high δ, and $q > q^n$ for all $n \geq 2$ and $q \in (0,1)$. □

[10] The equilibrium conditions are thus given by a system of linear equations, whose solution is presented in Appendix E.3.

And finally, we show that the solution to $\mathcal{RM}$-3 is also a solution to the revenue maximization problem $\mathcal{RM}$ for sufficiently high values of the buyers' discount factor:

Lemma 15. *Suppose that* $q \geq 1 - \frac{q^{n-1}(1-(1-q)^n)[n(1-q)-(1-q^n)]}{1-q^n}$. *Suppose further that* $\underline{b}^*$, $\overline{b}^*$, *and* v^*_{fse} *are as defined in (13), (14), and (15) respectively, then there exists a critical buyers' discount factor* δ^*, *such that for all* $\delta \in (\delta^*, 1)$ *and for all* $\delta_0 \in (0,1)$ *the tuple* $\left(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}\right)$ *solves* $\mathcal{RM}$, *implying in turn that the high-revenue strategy profile corresponding to* $(\overline{b}^*, \underline{b}^*)$ *is a collusive public perfect equilibrium of the repeated auction game. Moreover, the seller achieves full surplus extraction in the limit as* δ *goes to* 1.

Proof sketch. The complete proof is provided in Appendix G.3. As in the previous two cases, I only provide a sketch of the most important points in the main text. The argument for full surplus extraction is again the same: from (15) we have $nv^*_{\text{fse}} \xrightarrow[\delta\to 1]{} 0$, implying $\mathcal{R}^*_{\text{fse}}(\delta) \xrightarrow[\delta\to 1]{} (1-q^n)\overline{\theta} + q^n\underline{\theta}$. As in the previous two cases, full surplus extraction in the limit implies that $\mathcal{R}^*_{\text{fse}}(\delta) \geq (1-q^n)\overline{\theta}$ for all high enough values of δ, hence Proposition 1 and Lemma 14 imply that the high-revenue strategy profile corresponding to $\left(\overline{b}^*, \underline{b}^*\right)$ is a collusive public perfect equilibrium as long as the remaining constraints in $\mathcal{RM}$ are satisfied at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$ for high enough δ. The remaining constraints are checked by direct calculation.

The off-schedule incentive compatibility constraints (HighIC-up) and (HighIC-down) are satisfied by construction. Let us check that the low-type incentive compatibility constraint (LowIC) is satisfied. Recall that (LowIC), evaluated at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$, is given by:

$$(1-\delta)\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}^*) + \delta v^*_{\text{fse}} \geq 0.$$

Plugging the respective expressions from (13) and (15) into the above inequality, we obtain:

$$\delta \leq \frac{1}{1-q} - \frac{1-q^n}{n(1-q)^2},$$

which is true since $\frac{1}{1-q} - \frac{1-q^n}{n(1-q)^2} \geq 1$ because $q \geq 1 - \frac{q^{n-1}(1-(1-q)^n)[n(1-q)-(1-q^n)]}{1-q^n} > \frac{1-q^n}{n(1-q)}$.

The remaining constraints of $\mathcal{RM}$ are all non-binding at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$ for high values of δ and all values of q and n, except for the collusiveness constraint (Col-sep-1). Recall that (Col-sep-1) evaluated at $(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}})$ is given by:

$$v^*_{\text{fse}} \geq \frac{(1-\delta)(1-q^n)(\overline{\theta} - \underline{b}^*)}{n\big(1-\delta(1-q)^n\big)} = \frac{(1-\delta)(1-q^n)\delta q(1-q)(\overline{\theta} - \underline{\theta})}{nD(\delta)\big(1-\delta(1-q)^n\big)}.$$

Plugging the value of v^*_{fse} from (15) into the above expression, I obtain:

$$\big(1-\delta(1-q)^n\big)q^{n-1}\big[n(1-q)-(1-q^n)\big] \geq \delta(1-q^n)(1-q),$$

which can be satisfied for any $\delta \in (0,1)$ as long as q and n satisfy

$$q \geq 1 - \frac{q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]}{1-q^n},$$

which is true in Case 3 by assumption.

The rest of the constraints are checked in Appendix G.3. □

8 Revenue-maximizing reserve prices

The reserve prices along the equilibrium path of the full-surplus-extracting *collusive public perfect equilibria* (in the limit as δ goes to 1) are given by:

$$r^* = \begin{cases} \underline{\theta} + \frac{q(1-q^n)(\overline{\theta}-\underline{\theta})}{q(1-q^n)+q^n(1-(1-q)^n)} & \text{in Case 1} \\ \underline{\theta} + \frac{(1-q^n)^2(1-q)(\overline{\theta}-\underline{\theta})}{q^n(1-(1-q)^n)[n(1-q)-(1-q^n)]+(1-q^n)^2(1-q)} & \text{in Case 2} \\ \underline{\theta} + \frac{(1-q^n)(\overline{\theta}-\underline{\theta})}{1-q^n+q} & \text{in Case 3} \end{cases}$$

They are illustrated by Figure 4. The reserve prices are decreasing in q, going to $\overline{\theta}$ as q goes to 0 and going to $\underline{\theta}$ as q goes to 1. Indeed, since q is the probability of the low type, when q is close to 0, the buyers all have high valuations with a very high probability, and when q is close to 1, the buyers all have low valuations with a very high probability. Recall that the optimal reserve prices in the one-shot auction problem are also decreasing in q, but the optimal decision is essentially a cutoff rule (for fixed values of other parameters): for relatively low values of q the optimal reserve price is $\overline{\theta}$, while for relatively high values of q it is $\underline{\theta}$. Thus, even though the direction of dependence is the same, the functional form of this dependence is much less trivial in the repeated auction setting with collusion.

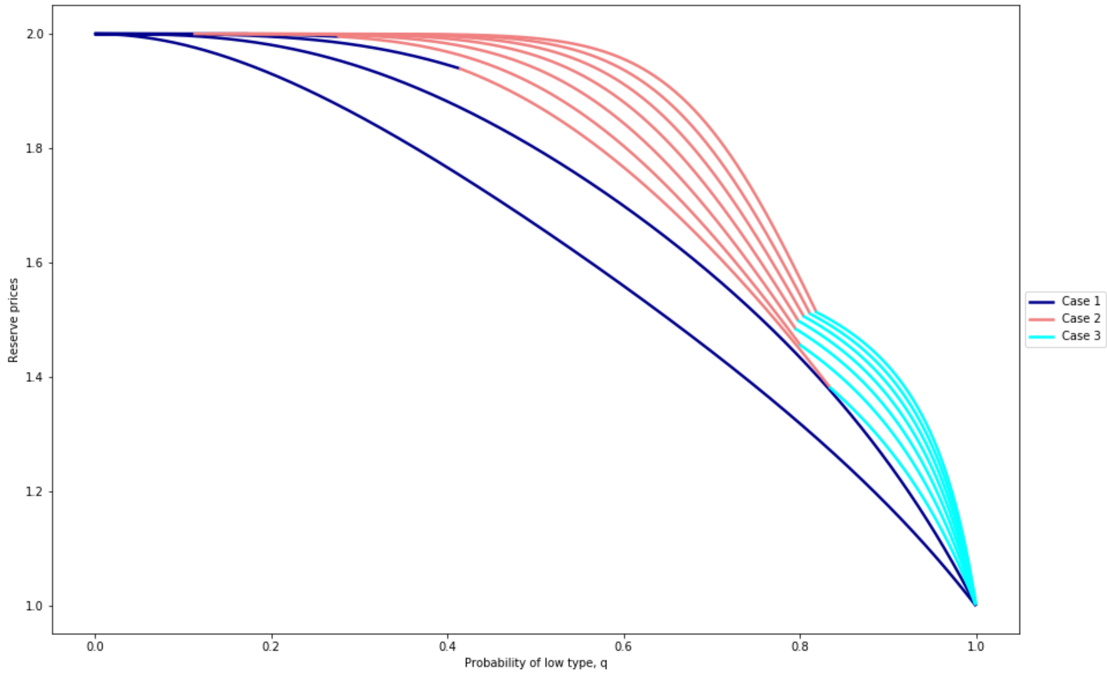


Figure 4: *Limiting reserve prices for all* $q \in (0, 1)$*, and for each* $n \in \{2, ..., 10\}$ *moving from the southwest to the northeast as* n *grows. Valuations are* $\underline{\theta} = 1$ *and* $\overline{\theta} = 2$*. In the dark-blue, red, and light-blue segments, Cases 1, 2, and 3 apply respectively.*

Similarly, the optimal reserve prices in the one-shot auction problem are increasing in the number of buyers, but the dependence takes the form of a cutoff rule (again, when the other parameter values are fixed), where the optimal reserve price is equal to $\underline{\theta}$ when the number of buyers is relatively low, and is equal to $\overline{\theta}$ when it is relatively high. In contrast to the one-shot setting, the reserve prices in the repeated auction setting with collusion depend on n in a much less trivial way.

This non-trivial dependence of the reserve prices on q and n can to a certain extent be explained by their very different role in the repeated setting with collusion. In the one-shot auction problem, the role of the reserve prices is to exclude certain valuation types from participation with the purpose of increasing competition among the remaining types. In the repeated setting with colluding buyers, the full-surplus-extracting *collusive public perfect equilibria* are efficient and the reserve prices play two crucial roles. First, in the off-path component of the seller's strategy, the reserve prices are chosen to punish the buyers for deviating from the equilibrium path bidding. Second, and more importantly, the on-path component of the reserve prices makes sure that the buyers pay "upfront" for the continuation of favorable terms of trade and at the same time do not have an incentive to collude on a lower bidding profile, resolving the fundamental conflict between revenue-maximization and fighting collusion.

9 Concluding remarks

In my paper, I have considered a repeated first-price auction model with a non-committed seller who dynamically adjusts reserve prices to fight collusion among buyers. To model the interaction between the seller and the colluding buyers, I have proposed the solution concept of *collusive public perfect equilibrium*. A collusive public perfect equilibrium is a public perfect equilibrium that additionally requires that the buyers be unable to improve their equilibrium payoff in the "buyer-game" induced by the seller's equilibrium strategy. Studying the outcomes as the buyers' discount factor goes to 1, I find collusive public perfect equilibria that allow the seller to extract the entire surplus from the colluding buyers. This result suggests that the problem of collusion in repeated auctions is perhaps less severe than is commonly understood: it turns out that a sufficiently sophisticated seller can come up with rather effective strategies for fighting collusion, even when she has to publicly disclose all the bids in the end of every period.

The buyers in my paper have access to symmetric collusive schemes. Such collusive schemes are particularly simple and thus might require no explicit communication among the buyers in practice, which makes them virtually impossible to detect for an antitrust authority. These hard-to-detect collusive schemes must therefore be addressed as part of the repeated auction design problem itself. My results imply that it can be done quite successfully. It is however well-known (see e.g. Mailath and Samuelson (2006)) that more sophisticated asymmetric collusive schemes might allow the buyers to collude more effectively, especially when they can communicate before the start of each auction. Even though such asymmetric collusive schemes can often dealt with by conventional means of antitrust policy, it is worth studying if they could also be addressed via more sophisticated auction design.

A Proof of Proposition 0

Proof. Let us first consider the choices made by the buyers who face a reserve price r. Depending on the reserve price chosen by the seller, there are three possible kinds of continuation games to consider:

Case I: $r \leq \underline{\theta}$

In this case, the following lemma applies:

Lemma 16. *If $r \leq \underline{\theta}$,*

- *any low-type buyer bids his own valuation in equilibrium: $\underline{b} = \underline{\theta}$,*
- *any high-type buyer randomizes his bids on $\left(\underline{\theta}, (1-q^{n-1})\overline{\theta} + q^{n-1}\underline{\theta}\right]$ according to*

$$G(b) = \frac{q}{1-q}\left[\left(\frac{\overline{\theta}-\underline{\theta}}{\overline{\theta}-b}\right)^{\frac{1}{n-1}} - 1\right].$$

The ex ante equilibrium payoff of the buyers is:

$$v^*_{r\leq\underline{\theta}} = (1-q)q^{n-1}(\overline{\theta}-\underline{\theta}).$$

The seller's expected revenue is:

$$\mathcal{R}^*_{r\leq\underline{\theta}} = (1-q^n)\overline{\theta} + q^n\underline{\theta} - n(1-q)q^{n-1}(\overline{\theta}-\underline{\theta}).$$

Proof. It is clear that both types will be willing to participate. It can be easily shown that there is no Nash equilibrium in pure strategies. It is also immediately clear that a low-type buyer will never place a bid higher than his own valuation because winning with such a high bid will lead to a negative payoff. But low types will not place a bid that is lower than their valuation even if they have an opportunity to do so. Suppose low-type buyers do place a bid $r < \underline{b} < \underline{\theta}$ in equilirium, then one of them could deviate to $\underline{b} + \epsilon$ and guarantee winning the auction for sure if his competitors are low types as well, hence there is a profitable deviation.

Suppose $\Phi(b)$ is the unconditional distribution of equilibrium bids for every buyer. The expected payoff of a buyer with type $\overline{\theta}$ is given by: $\Phi^{n-1}(b)(\overline{\theta}-b)$. Only low types bid $\underline{\theta}$, hence $\Phi(\underline{\theta}) = q$. By indifference we have:

$$\Phi^{n-1}(b)(\overline{\theta}-b) = q^{n-1}(\overline{\theta}-\underline{\theta}),$$

which means that $\Phi(b) = q\left(\frac{\overline{\theta}-\underline{\theta}}{\overline{\theta}-b}\right)^{\frac{1}{n-1}}$ and $v^*_{r\leq\underline{\theta}} = (1-q)q^{n-1}(\overline{\theta}-\underline{\theta})$.

To find the upper bound of the support we solve $q\left(\frac{\overline{\theta}-\underline{\theta}}{\overline{\theta}-\overline{b}}\right)^{\frac{1}{n-1}} = 1$, which leads to $\overline{b} = (1-q^{n-1})\overline{\theta} + q^{n-1}\underline{\theta}$. Since $\Phi(b)$ is the unconditional distribution of equilibrium bids, the actual mixed strategy of high-type buyers is given by:

$$G(b) \equiv \Phi(b|\theta_i = \overline{\theta}) = \frac{q}{1-q}\left[\left(\frac{\overline{\theta}-\underline{\theta}}{\overline{\theta}-b}\right)^{\frac{1}{n-1}} - 1\right].$$

The equilibrium is efficient, hence it leads to the total surplus given by $(1-q^n)\overline{\theta} + q^n\underline{\theta}$. The resulting revenue of the seller is

$$\begin{aligned}\mathcal{R}^*_{r\leq\underline{\theta}} &= (1-q^n)\overline{\theta} + q^n\underline{\theta} - nv^*_{r\leq\underline{\theta}} \\ &= (1-q^n)\overline{\theta} + q^n\underline{\theta} - n(1-q)q^{n-1}(\overline{\theta}-\underline{\theta}).\end{aligned}$$

□

Case II: $\underline{\theta} < r < \overline{\theta}$

In this case, the following lemma applies:

Lemma 17. *If* $\underline{\theta} < r < \overline{\theta}$,

- *any low-type buyer chooses to abstain:* $\underline{b} = \emptyset$,
- *any high-type buyer randomizes his bids on* $\left[r, (1-q^{n-1})\overline{\theta} + q^{n-1}r\right]$ *according to*

$$G(b) = \frac{q}{1-q}\left[\left(\frac{\overline{\theta}-r}{\overline{\theta}-b}\right)^{\frac{1}{n-1}} - 1\right].$$

The ex ante equilibrium payoff of the buyers is:

$$v^*_{\underline{\theta}<r<\overline{\theta}} = (1-q)q^{n-1}(\overline{\theta}-r).$$

The expected revenue of the seller is:

$$\mathcal{R}^*_{\underline{\theta}<r<\overline{\theta}} = (1-q^n)\overline{\theta} - n(1-q)q^{n-1}(\overline{\theta}-r).$$

Proof. In this case, only the high-type buyers are willing to participate. It can also be shown that there is no equilibrium in pure strategies. Hence we will be looking for an equilibrium in mixed strategies. Suppose that a high type buyer randomizes his bids according to the distribution function $G(b)$. The payoff of a high type buyer who is bidding b is given by:

$$\begin{aligned}&\big(q^{n-1} + (n-1)(1-q)q^{n-2}G(b) + ... + (1-q)^{n-1}G^{n-1}(b)\big)(\overline{\theta} - b)\\&= \big(q + (1-q)G(b)\big)^{n-1}(\overline{\theta} - b).\end{aligned}$$

Assuming that r is the lower bound of the support of $G(b)$ and that $G(b)$ has no mass points, we get $G(r) = 0$. By indifference, we get for every b in the support:

$$\big(q + (1-q)G(b)\big)^{n-1}(\overline{\theta} - b) = \big(q + (1-q)G(r)\big)^{n-1}(\overline{\theta} - r) = q^{n-1}(\overline{\theta} - r),$$

which immediately gives us

$$G(b) = \frac{q}{1-q}\left[\left(\frac{\overline{\theta} - r}{\overline{\theta} - b}\right)^{\frac{1}{n-1}} - 1\right],$$

and

$$v^*_{\underline{\theta}<r<\overline{\theta}} = (1-q)q^{n-1}(\overline{\theta} - r).$$

To find the upper bound of the support $\overline{b}$ we solve $\frac{q}{1-q}\big[\big(\frac{\overline{\theta}-r}{\overline{\theta}-\overline{b}}\big)^{\frac{1}{n-1}} - 1\big] = 1$ which leads to $\overline{b} = (1-q^{n-1})\overline{\theta} + q^{n-1}r$.

Since only the high-type buyers trade with the seller, the resulting total surplus is given by: $(1-q^n)\overline{\theta}$. The resulting revenue of the seller is then given by:

$$\mathcal{R}^*_{\underline{\theta}<r<\overline{\theta}} = (1-q^n)\overline{\theta} - nv^*_{\underline{\theta}<r<\overline{\theta}} = (1-q^n)\overline{\theta} - n(1-q)q^{n-1}(\overline{\theta} - r).$$

□

Case III: $r = \overline{\theta}$

In this case only high types are willing to participate, and they of course have no choice but to bid $b = \overline{\theta}$ in equilibrium, and the resulting revenue will be:

$$\mathcal{R}^*_{r=\overline{\theta}} = (1-q^n)\overline{\theta}.$$

Setting $r > \overline{\theta}$ cannot be an equilibrium strategy. Also, revenue achieved in **Case II** is inferior to that achieved in **Case III**, so setting $\underline{\theta} < r < \overline{\theta}$ cannot be an equilibrium strategy either.

□

B Proof of Lemma 4

Proof. Consider first the high-reserve-price state ω^h. Clearly in any public perfect equilibrium the payoff in this state must be zero, hence we can without loss of generality assume that $b_{\omega^h}(\overline{\theta}) = \overline{\theta}$ and $b_{\omega^h}(\underline{\theta}) = \emptyset$. Consider now the low-reserve-price state ω^l, in which the buyer-game starts. Consider any strongly symmetric public perfect equilibrium of the buyer-game. Pick any history that leads to ω^l and suppose any high-type buyer bids according to $b_{\omega^l}(\overline{\theta}) = \overline{b}$ and any low-type buyer bids according to $b_{\omega^l}(\underline{\theta}) = \underline{b}$ after that history, and the equilibrium continuation value is given by $v^*_{\omega^l} : A^n \to \mathbb{R}$. The equilibrium payoff of a high-type buyer i is given then by:

$$(1-\delta)p(\overline{b})(\overline{\theta}-\overline{b}) + \delta\,\mathbb{E}\left(v^*_{\omega^l}(\overline{b}, b_{\omega^l}(\theta_{-i}))\right),$$

where $p(\overline{b})$ is the winning probability from bidding $\overline{b}$ in the current period. Analogously the equilibrium payoff of a low-type buyer i is equal to:

$$(1-\delta)p(\underline{b})(\underline{\theta}-\underline{b}) + \delta\,\mathbb{E}\left(v^*_{\omega^l}(\underline{b}, b_{\omega^l}(\theta_{-i}))\right),$$

where $p(\underline{b})$ is the winning probability from bidding $\underline{b}$ in the current period.

Since the above are public perfect equilibrium payoffs, the following incentive compatibility constraints must be satisfied, for a high type buyer:

$$(1-\delta)p(\overline{b})(\overline{\theta}-\overline{b}) + \delta\,\mathbb{E}\left(v^*_{\omega^l}(\overline{b}, b_{\omega^l}(\theta_{-i}))\right) \geq (1-\delta)p(\underline{b})(\overline{\theta}-\underline{b}) + \delta\,\mathbb{E}\left(v^*_{\omega^l}(\underline{b}, b_{\omega^l}(\theta_{-i}))\right), \quad (16)$$

and for a low type buyer:

$$(1-\delta)p(\underline{b})(\underline{\theta}-\underline{b}) + \delta\,\mathbb{E}\left(v^*_{\omega^l}(\underline{b}, b_{\omega^l}(\theta_{-i}))\right) \geq (1-\delta)p(\overline{b})(\underline{\theta}-\overline{b}) + \delta\,\mathbb{E}\left(v^*_{\omega^l}(\overline{b}, b_{\omega^l}(\theta_{-i}))\right). \quad (17)$$

Adding (16) and (17) and canceling the continuation values on both sides, we get:

$$\begin{aligned}
&(1-\delta)p(\overline{b})(\overline{\theta}-\overline{b}) + (1-\delta)p(\underline{b})(\underline{\theta}-\underline{b}) \geq (1-\delta)p(\underline{b})(\overline{\theta}-\underline{b}) + (1-\delta)p(\overline{b})(\underline{\theta}-\overline{b})\\
&\Leftrightarrow p(\overline{b})(\overline{\theta}-\overline{b}) + p(\underline{b})(\underline{\theta}-\underline{b}) \geq p(\underline{b})(\overline{\theta}-\underline{b}) + p(\overline{b})(\underline{\theta}-\overline{b})\\
&\Leftrightarrow p(\overline{b})\overline{\theta} + p(\underline{b})\underline{\theta} \geq p(\underline{b})\overline{\theta} + p(\overline{b})\underline{\theta}\\
&\Leftrightarrow \left(p(\overline{b}) - p(\underline{b})\right)(\overline{\theta}-\underline{\theta}) \geq 0\\
&\Leftrightarrow p(\overline{b}) - p(\underline{b}) \geq 0,
\end{aligned}$$

which establishes the claim. □

C Proof of Lemma 6

Proof. [11] Let $\mathcal{V}$ denote the set of strongly symmetric public perfect equilibrium payoffs of the low-revenue buyer-game in Definition 10 satisfying Assumption 1(b), and let $\hat{v} = \sup \mathcal{V}$. We distinguish two classes of equilibria: those in which a separating bidding profile is played *in the first period*, and those in which a pooling bidding profile is played *in the first period*. Let $\mathcal{V}_{\text{sep}}$ and $\mathcal{V}_{\text{pool}}$ be the corresponding sets of strongly symmetric public perfect equilibrium payoffs. Clearly $\mathcal{V} = \mathcal{V}_{\text{sep}} \cup \mathcal{V}_{\text{pool}}$, hence $\hat{v} = \sup \mathcal{V}_{\text{sep}}$ or $\hat{v} = \sup \mathcal{V}_{\text{pool}}$.

Separation in the first period. Suppose $\hat{v} = \sup \mathcal{V}_{\text{sep}}$ and consider a strongly symmetric public perfect equilibrium in which the buyers separate in the first period. Let $b : \Theta \to A$ be the equilibrium action taken in the first period. Denote $\underline{b}$ and $\bar{b}$ the bids placed in the first period by a low-type buyer and a high-type buyer respectively. Suppose that the equilibrium continuation value after the first period is given by $v^* : A^n \to \mathbb{R}$, then the equilibrium payoff of a high-type buyer i is given by:

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\bar{\theta} - \bar{b}) + \delta\, \mathbb{E}\left(v^*(\bar{b}, b(\theta_{-i}))\right).$$

The equilibrium payoff of a low-type buyer i is given by:

$$(1-\delta)\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}) + \delta\, \mathbb{E}\left(v^*(\underline{b}, b(\theta_{-i}))\right).$$

The on-schedule incentive compatibility constraint of a low-type buyer is then given by:

$$(1-\delta)\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}) + \delta\, \mathbb{E}\left(v^*(\underline{b}, b(\theta_{-i}))\right) \geq (1-\delta)\frac{1-q^n}{n(1-q)}(\underline{\theta} - \bar{b}) + \delta\, \mathbb{E}\left(v^*(\bar{b}, b(\theta_{-i}))\right).$$

Subtract $\delta\hat{v}$ and divide both sides by $(1-\delta)$:

$$\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}) + \frac{\delta}{1-\delta}\mathbb{E}\left(v^*(\underline{b}, b(\theta_{-i})) - \hat{v}\right) \geq \frac{1-q^n}{n(1-q)}(\underline{\theta} - \bar{b}) + \frac{\delta}{1-\delta}\mathbb{E}\left(v^*(\bar{b}, b(\theta_{-i})) - \hat{v}\right),$$

and define $\bar{x} \equiv \frac{\delta}{1-\delta}\mathbb{E}\left(v^*(\bar{b}, b(\theta_{-i})) - \hat{v}\right)$ and $\underline{x} \equiv \frac{\delta}{1-\delta}\mathbb{E}\left(v^*(\underline{b}, b(\theta_{-i})) - \hat{v}\right)$. The incentive compatibility constraint of a low-type buyer can then be written as:

$$\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}) + \underline{x} \geq \frac{1-q^n}{n(1-q)}(\underline{\theta} - \bar{b}) + \bar{x}. \tag{18}$$

[11] See a similar argument in Chapter 11.2 of Mailath and Samuelson (2006) in the context of a repeated price competition game with adverse selection.

Observe that for any strongly symmetric public perfect equilibrium satisfying Assumption 1(b), the continuation equilibria after any public history *consistent with the on-path play of the equilibrium strategies* are themselves strongly symmetric public perfect equilibria satisfying Assumption 1(b), hence we must have $\overline{x} \leq 0$ and $\underline{x} \leq 0$ since $\hat{v} = \sup \mathcal{V}$.

The *ex ante* equilibrium payoff is given by:

$$(1-\delta)\frac{1}{n}\left[(1-q^n)(\overline{\theta}-\overline{b}) + q^n(\underline{\theta}-\underline{b})\right] + (1-q)\delta\,\mathbb{E}\left(v^*(\overline{b}, b(\theta_{-i}))\right) + q\delta\,\mathbb{E}\left(v^*(\underline{b}, b(\theta_{-i}))\right).$$

Subtracting $\delta\hat{v}$ and dividing by $(1-\delta)$, we obtain:

$$\frac{1}{n}\left[(1-q^n)(\overline{\theta}-\overline{b}) + q^n(\underline{\theta}-\underline{b})\right] + (1-q)\frac{\delta}{1-\delta}\,\mathbb{E}\left(v^*(\overline{b}, b(\theta_{-i})) - \hat{v}\right) + q\frac{\delta}{1-\delta}\,\mathbb{E}\left(v^*(\underline{b}, b(\theta_{-i})) - \hat{v}\right),$$

which can be rewritten as:

$$\frac{1}{n}\left[(1-q^n)(\overline{\theta}-\overline{b}) + q^n(\underline{\theta}-\underline{b})\right] + (1-q)\overline{x} + q\underline{x}.$$

Combining this expression with the low-type incentive compatibilty constraint in (18) and our observation that $\underline{x}, \overline{x} \leq 0$, we must conclude that[12]:

$$\hat{v} = \frac{\hat{v}-\delta\hat{v}}{1-\delta} = \sup_{v\in\mathcal{V}_{\text{sep}}} \frac{v-\delta\hat{v}}{1-\delta} \leq \sup_{\underline{b},\overline{b};\underline{x},\overline{x}} \frac{1}{n}\left[(1-q^n)(\overline{\theta}-\overline{b}) + q^n(\underline{\theta}-\underline{b})\right] + (1-q)\overline{x} + q\underline{x} \tag{19}$$
$$\text{subject to} \quad (\text{IC}) \quad \frac{q^{n-1}}{n}(\underline{\theta}-\underline{b}) + \underline{x} \geq \frac{1-q^n}{n(1-q)}(\underline{\theta}-\overline{b}) + \overline{x},$$
$$(\text{Feas}) \quad \underline{x}, \overline{x} \leq 0.$$

Let us consider the maximization problem in (19). Clearly the (IC) constraint must be binding at an optimum: suppose not, then choose $\overline{b}' < \overline{b}$ such that the constraint is still satisfied, and this will clearly improve the value of the objective. Hence, at any optimum of (19), we must have

$$\frac{q^{n-1}}{n}(\underline{\theta}-\underline{b}) + \underline{x} = \frac{1-q^n}{n(1-q)}(\underline{\theta}-\overline{b}) + \overline{x},$$

which we can solve for $(1-q^n)(\underline{\theta}-\overline{b})$, to obtain:

$$(1-q^n)(\underline{\theta}-\overline{b}) = (1-q)q^{n-1}(\underline{\theta}-\underline{b}) + n(1-q)(\underline{x}-\overline{x}),$$

[12] The solution to this maximization problem provides an upper bound on (normalized) strongly symmetric equilibrium payoffs since all the other incentive compatibility constraints are ignored, and the constraint $\underline{x}, \overline{x} \leq 0$ is necessary for feasibility of continuation values but not sufficient.

which then implies:

$$(1-q^n)(\overline{\theta}-\overline{b}) = (1-q^n)(\overline{\theta}-\underline{\theta}) + (1-q)q^{n-1}(\underline{\theta}-\underline{b}) + n(1-q)(\underline{x}-\overline{x}). \tag{20}$$

Plugging (20) into the objective function in (19), we get:

$$\frac{1}{n}\big[(1-q^n)(\overline{\theta}-\underline{\theta}) + (1-q)q^{n-1}(\underline{\theta}-\underline{b}) + n(1-q)(\underline{x}-\overline{x}) + q^n(\underline{\theta}-\underline{b})\big] + (1-q)\overline{x} + q\underline{x}$$
$$= \frac{1}{n}(1-q^n)(\overline{\theta}-\underline{\theta}) + \frac{1}{n}q^{n-1}(\underline{\theta}-\underline{b}) + \underline{x},$$

which implies that:

$$\hat{v} \le \sup_{\underline{b},\underline{x}} \frac{1}{n}(1-q^n)(\overline{\theta}-\underline{\theta}) + \frac{1}{n}q^{n-1}(\underline{\theta}-\underline{b}) + \underline{x} \;\text{ subject to } \overline{x} \le 0.$$

The optimum is clearly achieved when $\underline{b}=0$ and $\underline{x}=0$, which means that:

$$\hat{v} \le \frac{1}{n}(1-q^n)(\overline{\theta}-\underline{\theta}) + \frac{1}{n}q^{n-1}\underline{\theta} = v_{\text{lrs}}.$$

Pooling in the first period. Suppose $\hat{v} = \sup \mathcal{V}_{\text{pool}}$ and consider a strongly symmetric public perfect equilibrium in which the buyers separate in the first period, and let b be the equilibrium action of both types in the first period. Suppose that $v^* : A^n \to \mathbb{R}$ is the equilibrium continuation value after the first period. The *ex ante* equilibrium payoff is:

$$(1-\delta)\frac{1}{n}\big((1-q)\overline{\theta} + q\underline{\theta} - b\big) + \delta v^*(b,\ldots,b)$$

Subtracting $\delta\hat{v}$ and dividing by $(1-\delta)$, we obtain:

$$\frac{1}{n}\big((1-q)\overline{\theta} + q\underline{\theta} - b\big) + \frac{\delta}{1-\delta}(v^*(b,\ldots,b) - \hat{v})$$

Denote $x \equiv \frac{\delta}{1-\delta}(v^*(b,\ldots,b)-\hat{v})$. As in the previous case, we have $x \le 0$, and therefore:

$$\hat{v} = \frac{\hat{v}-\delta\hat{v}}{1-\delta} = \sup_{v\in\mathcal{V}_{\text{pool}}} \frac{v-\delta\hat{v}}{1-\delta} \le \sup_{b,x} \frac{1}{n}\big((1-q)\overline{\theta} + q\underline{\theta} - b\big) + x \;\text{ subject to } x \le 0$$
$$\le \frac{1}{n}\big[(1-q)\overline{\theta} + q\underline{\theta}\big] = v_{\text{zrp}}$$

The direct comparison of v_{lrs} and v_{zrp} establishes the result. □

D Proofs of Lemmas 7, 10, 13

(Existence of solutions to the relaxed programs)

D.1 Proof of Lemma 7

Proof. $\mathcal{RM}$-1 is a linear programming problem. If it has a solution, then it also has solution at a vertex of its feasible set. The feasible set is non-empty and has a unique vertex, at which (Col-sep-1) and (LowIC) are binding. It therefore remains to rule out the situation, in which the value of $\mathcal{RM}$-1 is unbounded. The Duality Theorem of Linear Programming (see e.g. Section 3.2 in Luenberger and Ye (2021)) implies that if the linear programming dual of $\mathcal{RM}$-1 is feasible, then the value of $\mathcal{RM}$-1 is bounded. The dual of $\mathcal{RM}$-1 is:

$$
\begin{aligned}
&\text{Dual-1}: \quad \max_{\lambda_{\text{IC}},\lambda_{\text{Col}}} \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{\theta})}{1-\delta(1-q)^n}\lambda_{\text{Col}}, \quad \text{s.t.} \\
&\text{(I)} \qquad \left[(1-\delta)q^{n-1}+\delta q^n\right]\lambda_{\text{IC}} + \left[q^n + \frac{(1-\delta)(1-q^n)}{1-\delta(1-q)^n}\right]\lambda_{\text{Col}} = q^n, \\
&\text{(II)} \qquad \delta\lambda_{\text{IC}} + \lambda_{\text{Col}} = 1, \\
&\text{(DF)} \qquad \lambda_{\text{IC}} \geq 0, \ \lambda_{\text{Col}} \geq 0.
\end{aligned}
$$

Direct calculation shows that:

$$
\begin{aligned}
\lim_{\delta\to 1} \lambda_{\text{IC}}(\delta) &= \frac{q(1-q^n)}{q(1-q^n)+q^n\big(1-(1-q)^n\big)} > 0, \\
\lim_{\delta\to 1} \lambda_{\text{Col}}(\delta) &= \frac{q\big(1-(1-q)^n\big)}{q(1-q^n)+q^n\big(1-(1-q)^n\big)} > 0,
\end{aligned}
$$

which implies that the dual of $\mathcal{RM}$-1 is feasible for all sufficiently high δ. □

D.2 Proof of Lemma 10

Proof. $\mathcal{RM}$-2 is a linear programming problem. If it has a solution, then it also has solution at a vertex of its feasible set. The feasible set is non-empty and has a unique vertex, at which (Col-sep-1) and (HighIC-up) are binding. It therefore remains to rule out the situation, in which the value of $\mathcal{RM}$-2 is unbounded. The Duality Theorem of Linear Programming (see e.g. Section 3.2 in Luenberger and Ye (2021)) implies that if the linear programming dual of $\mathcal{RM}$-2 is feasible, then the value of $\mathcal{RM}$-2 is bounded. The dual of $\mathcal{RM}$-2 is:

$$\text{Dual-2}: \quad \max_{\lambda_{\text{IC}},\lambda_{\text{Col}}} \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{\theta})}{1-\delta(1-q)^n}\lambda_{\text{Col}}, \quad \text{s.t.}$$

$$\text{(I)} \qquad \delta q^n \lambda_{\text{IC}} + \left[q^n - \frac{(1-\delta)(1-q^n)}{1-\delta(1-q)^n}\right]\lambda_{\text{Col}} = q^n,$$

$$\text{(II)} \qquad \left[(1-\delta)\frac{1-q^n}{1-q} + \delta(1-q^n) - n(1-\delta)\right]\lambda_{\text{IC}} + (1-q^n)\lambda_{\text{Col}} = 1-q^n,$$

$$\text{(DF)} \qquad \lambda_{\text{IC}} \geq 0, \ \lambda_{\text{Col}} \geq 0.$$

Direct calculation shows that:

$$\lim_{\delta\to 1} \lambda_{\text{IC}}(\delta) = \frac{(1-q)\big(1-q^n\big)^2}{q^n\big(1-(1-q)^n\big)\big(n(1-q)-(1-q^n)\big)+(1-q)\big(1-q^n\big)^2} > 0,$$

$$\lim_{\delta\to 1} \lambda_{\text{Col}}(\delta) = \frac{q^n\big(1-(1-q)^n\big)\big(n(1-q)-(1-q^n)\big)}{q^n\big(1-(1-q)^n\big)\big(n(1-q)-(1-q^n)\big)+(1-q)\big(1-q^n\big)^2} > 0,$$

which implies that the dual of $\mathcal{RM}$-2 is feasible for all sufficiently high δ. □

D.3 Proof of Lemma 13

Proof. $\mathcal{RM}$-3 is a linear programming problem. If it has a solution, then it also has solution at a vertex of its feasible set. The feasible set is non-empty and has a unique vertex, at which (HighIC-down) and (HighIC-up) are binding. It therefore remains to rule out the situation, in which the value of $\mathcal{RM}$-3 is unbounded. The Duality Theorem of Linear Programming (see e.g. Section 3.2 in Luenberger and Ye (2021)) implies that if the linear programming dual of $\mathcal{RM}$-3 is feasible, then the value of $\mathcal{RM}$-3 is bounded. The dual of $\mathcal{RM}$-3 is:

$$\text{Dual-3}: \quad \max_{\lambda_{\text{up}},\lambda_{\text{down}}} (1-\delta)nq^{n-1}(\overline{\theta}-\underline{\theta})\lambda_{\text{down}}, \quad \text{s.t.}$$

$$\text{(I)} \qquad \delta q \lambda_{\text{up}} + \big[\delta q - n(1-\delta)\big]\lambda_{\text{down}} = q,$$

$$\text{(II)} \qquad \left[\frac{1-\delta q}{1-q} - \frac{n(1-\delta)}{1-q^n}\right]\lambda_{\text{up}} + \frac{1-\delta q}{1-q}\lambda_{\text{down}} = 1,$$

$$\text{(DF)} \qquad \lambda_{\text{up}} \geq 0, \ \lambda_{\text{down}} \geq 0.$$

Direct calculation shows that:

$$\lim_{\delta\to 1} \lambda_{\text{up}}(\delta) = \frac{(1-q^n)\big(n(1-q)+q\big)}{n(1-q)(1-q^n+q)} > 0,$$

$$\lim_{\delta\to 1} \lambda_{\text{down}}(\delta) = \frac{q\big(n(1-q)-(1-q^n)\big)}{n(1-q)(1-q^n+q)} > 0,$$

which implies that the dual of $\mathcal{RM}$-3 is feasible for all sufficiently high δ. □

E Solutions of equilibrium conditions

E.1 Solution of Case 1

Recall that the equilibrium conditions in Case 1 are:

$$v^*_{\text{fse}} = \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{n(1-\delta(1-q)^n)}, \tag{21}$$

$$(1-\delta)\frac{q^{n-1}}{n}(\underline{\theta}-\underline{b}^*)+\delta v^*_{\text{fse}} = 0, \tag{22}$$

$$v^*_{\text{fse}} = \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\big]. \tag{23}$$

Combining the equations (21) and (22), we get

$$(1-\delta)\frac{q^{n-1}}{n}(\underline{\theta}-\underline{b}^*)+\delta\frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{n(1-\delta(1-q)^n)} = 0,$$

which we can solve for the equilibrium value of $\underline{b}$:

$$\underline{b}^* = \frac{\delta q\big(1-q^n\big)\overline{\theta}+q^n\big(1-\delta(1-q)^n\big)\underline{\theta}}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)},$$

which we can now use to compute the payoff of each type conditional upon winning with $\underline{b}^*$, for a low type buyer we have:

$$\begin{aligned}
\underline{\theta}-\underline{b}^* &= \underline{\theta}-\frac{\delta q\big(1-q^n\big)\overline{\theta}+q^n\big(1-\delta(1-q)^n\big)\underline{\theta}}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} \\
&= \frac{\delta q\big(1-q^n\big)\underline{\theta}+q^n\big(1-\delta(1-q)^n\big)\underline{\theta}-\delta q\big(1-q^n\big)\overline{\theta}-q^n\big(1-\delta(1-q)^n\big)\underline{\theta}}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} \\
&= \frac{-\delta q\big(1-q^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} < 0;
\end{aligned} \tag{24}$$

and for a high type buyer we have:

$$\begin{aligned}
\overline{\theta}-\underline{b}^* &= \overline{\theta}-\frac{\delta q\big(1-q^n\big)\overline{\theta}+q^n\big(1-\delta(1-q)^n\big)\underline{\theta}}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} \\
&= \frac{\delta q\big(1-q^n\big)\overline{\theta}+q^n\big(1-\delta(1-q)^n\big)\overline{\theta}-\delta q\big(1-q^n\big)\overline{\theta}-q^n\big(1-\delta(1-q)^n\big)\underline{\theta}}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} \\
&= \frac{q^n\big(1-\delta(1-q)^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} > 0,
\end{aligned} \tag{25}$$

which combined with (21) gives us the resulting equilibrium payoff:

$$v^*_{\text{fse}} = \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{n(1-\delta(1-q)^n)} = \tag{26}$$
$$= \frac{(1-\delta)(1-q^n)}{n(1-\delta(1-q)^n)} \times \frac{q^n\big(1-\delta(1-q)^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}$$
$$= \frac{1}{n}\frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}.$$

From (23) we have:

$$\frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\big] = \frac{1}{n}\frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)},$$

which, knowing $\underline{\theta}-\underline{b}^*$ from (24), we can solve for $\overline{\theta}-\overline{b}^*$ to obtain:

$$\overline{\theta}-\overline{b}^* = \frac{(1-\delta)q^n(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} - \frac{q^n(\underline{\theta}-\underline{b}^*)}{1-q^n} \tag{27}$$
$$= \frac{(1-\delta)q^n(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} + \frac{q^n\delta q(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}$$
$$= \frac{q^n(1-\delta(1-q))(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)},$$

We can now use expression (27) to determine $\overline{b}^*$:

$$\overline{b}^* = \overline{\theta} - \frac{q^n(1-\delta(1-q))(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}.$$

E.2 Solution of Case 2

Recall that in Case 2 the equilibrium conditions are given by:

$$v^*_{\text{fse}} = \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{n(1-\delta(1-q)^n)}, \tag{28}$$
$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*) + \delta v^*_{\text{fse}} = (1-\delta)(\overline{\theta}-\overline{b}^*), \tag{29}$$
$$v^*_{\text{fse}} = \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\big]. \tag{30}$$

The equilibrium condition in (28) implies that:

$$(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}) = \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{1-\delta(1-q)^n},$$

which can in turn be rewritten as:

$$(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*) = \frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{\theta})}{1-\delta(1-q)^n} + \frac{(1-\delta)(1-q^n)(\underline{\theta}-\underline{b})}{1-\delta(1-q)^n}.$$

Collecting terms, we get:

$$(1-q^n)(\overline{\theta}-\overline{b}^*)+\left[\frac{q^n-\delta q^n(1-q)^n-(1-\delta)(1-q^n)}{1-\delta(1-q)^n}\right](\underline{\theta}-\underline{b}^*)=\frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{\theta})}{1-\delta(1-q)^n}. \quad (31)$$

Recall that (29) implies

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)+\frac{\delta}{n}\left[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b})\right]=(1-\delta)(\overline{\theta}-\overline{b}^*).$$

This condition can be rewritten as:

$$\begin{aligned}\frac{\delta q^n}{n}(\underline{\theta}-\underline{b}^*)&=(1-\delta)(\overline{\theta}-\overline{b}^*)-(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)-\frac{\delta}{n}(1-q^n)(\overline{\theta}-\overline{b}^*)\\&=(1-\delta)(\overline{\theta}-\overline{b}^*)-\frac{1-q^n}{n}(\overline{\theta}-\overline{b}^*)\left(\frac{1-\delta}{1-q}+\delta\right)\\&=(\overline{\theta}-\overline{b}^*)\left[(1-\delta)-\frac{1-q^n}{n(1-q)}(1-\delta q)\right]\\&=\frac{\left[n(1-q)(1-\delta)-(1-q^n)(1-\delta q)\right](\overline{\theta}-\overline{b}^*)}{n(1-q)}. \quad (32)\end{aligned}$$

Using equations (31) and (32), we can write:

$$(1-q^n)(\overline{\theta}-\overline{b}^*)+\left[\frac{q^n-\delta q^n(1-q)^n-(1-\delta)(1-q^n)}{1-\delta(1-q)^n}\right](\underline{\theta}-\underline{b}^*)=\frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{\theta})}{1-\delta(1-q)^n},$$

$$\delta q^n(\underline{\theta}-\underline{b}^*)=\frac{\left[n(1-q)(1-\delta)-(1-q^n)(1-\delta q)\right](\overline{\theta}-\overline{b}^*)}{1-q},$$

which can be solved for optimal payoffs $\overline{\theta}-\overline{b}^*$ and $\underline{\theta}-\underline{b}^*$:

$$\begin{aligned}\underline{\theta}-\underline{b}^*&=-\frac{1}{D(\delta)}\left[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\right](1-q^n)(\overline{\theta}-\underline{\theta}),\\\overline{\theta}-\overline{b}^*&=\frac{1}{D(\delta)}\delta q^n(1-q^n)(1-q)(\overline{\theta}-\underline{\theta}),\end{aligned}$$

where $D(\delta)$ is given by:

$$D(\delta)=q^n\big(1-\delta(1-q)^n\big)\left[n(1-q)-(1-q^n)\right]+(1-q^n)\left[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\right].$$

The *ex ante* equilibrium payoff can be found from:

$$\begin{aligned}nv^*_{\text{fse}}&=(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\\&=\frac{q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{D(\delta)}\left[\delta(1-q^n)(1-q)-(1-q^n)(1-\delta q)+n(1-\delta)(1-q)\right]\\&=\frac{q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{D(\delta)}\left[(1-q^n)(\delta-\delta q-1+\delta q)+n(1-\delta)(1-q)\right]\\&=\frac{q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{D(\delta)}(1-\delta)\left[-(1-q^n)+n(1-q)\right].\end{aligned}$$

Hence the *ex ante* equilibrium payoff is:

$$v^*_{\text{fse}} = \frac{1}{nD(\delta)}(1-\delta)q^n(1-q^n)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}). \tag{33}$$

We can now determine the payoff of the high type who wins with a low bid, i.e. $\overline{\theta}-\underline{b}^*$. Combining the expression for the *ex ante* equilibrium payoff in (33) and the equilibrium condition in (28) we get

$$\frac{(1-\delta)(1-q^n)(\overline{\theta}-\underline{b}^*)}{n(1-\delta(1-q)^n)} = \frac{1}{nD(\delta)}(1-\delta)q^n(1-q^n)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}),$$

which can be solved for $\overline{\theta}-\underline{b}^*$:

$$\overline{\theta}-\underline{b}^* = \frac{1}{D(\delta)}q^n(1-\delta(1-q)^n)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}).$$

E.3 Solution of Case 3

Recall that in Case 3 the equilibrium conditions are given by:

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)+\delta v^*_{\text{fse}} = (1-\delta)(\overline{\theta}-\overline{b}^*), \tag{34}$$

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)+\delta v^*_{\text{fse}} = (1-\delta)q^{n-1}(\overline{\theta}-\underline{b}^*), \tag{35}$$

$$v^*_{\text{fse}} = \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\big]. \tag{36}$$

Note that conditions (34) and (35) together imply $\overline{\theta}-\overline{b}^* = q^{n-1}(\overline{\theta}-\underline{b}^*)$. Hence the equilibrium payoff becomes:

$$\begin{aligned}
v^*_{\text{fse}} &= \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\big]\\
&= \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\overline{\theta}-\overline{\theta}+\underline{\theta}-\underline{b}^*)\big]\\
&= \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\overline{\theta}-\underline{b}^*)-q^n(\overline{\theta}-\underline{\theta})\big]\\
&= \frac{1}{n}\big[(1-q^n)(\overline{\theta}-\overline{b}^*)+q(\overline{\theta}-\overline{b}^*)-q^n(\overline{\theta}-\underline{\theta})\big]\\
&= \frac{1}{n}\big[(1-q^n+q)(\overline{\theta}-\overline{b}^*)-q^n(\overline{\theta}-\underline{\theta})\big].
\end{aligned} \tag{37}$$

The upward incentive compatibility constraint in (34) can then be written as:

$$(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*)+\delta\frac{1}{n}\big[(1-q^n+q)(\overline{\theta}-\overline{b}^*)-q^n(\overline{\theta}-\underline{\theta})\big] = (1-\delta)(\overline{\theta}-\overline{b}^*).$$

which can then be solved for $\overline{\theta} - \overline{b}^*$:

$$\overline{\theta} - \overline{b}^* = \frac{\delta q^n (1-q)(\overline{\theta} - \underline{\theta})}{(1-q^n)(1-\delta q) + \delta q(1-q) - n(1-\delta)(1-q)}. \tag{38}$$

We can now introduce shorthand notation for the denominator:

$$D(\delta) = (1-q^n)(1-\delta q) + \delta q(1-q) - n(1-\delta)(1-q).$$

The *ex ante* equilibrium payoff can now be calculated from (37):

$$\begin{aligned}
nv^*_{\text{fse}} &= (1-q^n+q)(\overline{\theta} - \overline{b}^*) - q^n(\overline{\theta} - \underline{\theta}) \\
&= (1-q^n+q)\frac{\delta q^n (1-q)(\overline{\theta} - \underline{\theta})}{(1-q^n)(1-\delta q) + \delta q(1-q) - n(1-\delta)(1-q)} - q^n(\overline{\theta} - \underline{\theta}) \\
&= \frac{q^n(\overline{\theta} - \underline{\theta})}{D(\delta)}\big[(1-q^n+q)\delta(1-q) - (1-q^n)(1-\delta q) - \delta q(1-q) + n(1-\delta)(1-q)\big] \\
&= \frac{q^n(\overline{\theta} - \underline{\theta})}{D(\delta)}\big[(1-q^n)\big(\delta(1-q) - (1-\delta q)\big) + n(1-\delta)(1-q)\big] \\
&= \frac{(1-\delta)q^n(\overline{\theta} - \underline{\theta})}{D(\delta)}\big[n(1-q) - (1-q^n)\big] \\
&= \frac{1}{D(\delta)}(1-\delta)q^n\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}).
\end{aligned}$$

The *ex ante* equilibrium payoff is then given by:

$$v^*_{\text{fse}} = \frac{1}{nD(\delta)}(1-\delta)q^n\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}).$$

The payoff of a high type buyer who wins with the low bid can be calculated from (38) and the fact that $\overline{\theta} - \underline{b}^* = \frac{1}{q^{n-1}}(\overline{\theta} - \overline{b}^*)$, and is therefore given by:

$$\begin{aligned}
\overline{\theta} - \underline{b}^* &= \frac{\delta q(1-q)(\overline{\theta} - \underline{\theta})}{(1-q^n)(1-\delta q) + \delta q(1-q) - n(1-\delta)(1-q)} \\
&= \frac{1}{D(\delta)}\delta q(1-q)(\overline{\theta} - \underline{\theta}).
\end{aligned}$$

A low-type buyer payoff can be calculated from $nv^*_{\text{fse}} = (1-q^n)(\overline{\theta} - \overline{b}^*) + q^n(\underline{\theta} - \underline{b}^*)$:

$$\begin{aligned}
q^n(\underline{\theta} - \underline{b}^*) &= nv^*_{\text{fse}} - (1-q^n)(\overline{\theta} - \overline{b}^*) \\
&= \frac{1}{D(\delta)}(1-\delta)q^n\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}) - (1-q^n)\frac{1}{D(\delta)}\delta q^n(1-q)(\overline{\theta} - \underline{\theta}),
\end{aligned}$$

which implies:

$$\begin{aligned}
\underline{\theta} - \underline{b}^* &= \frac{1}{D(\delta)}\Big[(1-\delta)\big[n(1-q) - (1-q^n)\big] - (1-q^n)\delta(1-q)\Big](\overline{\theta} - \underline{\theta}) \\
&= \frac{1}{D(\delta)}\big[n(1-q)(1-\delta) - (1-q^n)(1-\delta q)\big](\overline{\theta} - \underline{\theta})
\end{aligned}$$

F Proof of Lemma 11

Proof. We have shown $\underline{\theta} < \underline{b}^*$ in the main text. To show $\underline{b}^* < \overline{b}^*$, it suffices to show that $\overline{\theta} - \underline{b}^* > \overline{\theta} - \overline{b}^*$. Observe that

$$\overline{\theta} - \underline{b}^* = \frac{1}{D(\delta)} q^n \big(1 - \delta(1-q)^n\big)\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}).$$

$\overline{\theta} - \underline{b}^* > \overline{\theta} - \overline{b}^*$ can then be written as:

$$\frac{1}{D(\delta)} q^n \big(1 - \delta(1-q)^n\big)\big[n(1-q) - (1-q^n)\big](\overline{\theta} - \underline{\theta}) > \frac{1}{D(\delta)} \delta q^n (1-q^n)(1-q)(\overline{\theta} - \underline{\theta})$$
$$\Leftrightarrow \big(1 - \delta(1-q)^n\big)\big[n(1-q) - (1-q^n)\big] > \delta(1-q^n)(1-q).$$

It is easy to see that the above inequality holds for all δ whenever:

$$\big(1 - (1-q)^n\big)\big[n(1-q) - (1-q^n)\big] > (1-q^n)(1-q). \tag{39}$$

Recall now that we assume that $q \geq \frac{1-q^n}{n(1-q)}$, which is equivalent to:

$$n(1-q)^2 q \geq (1-q^n)(1-q), \tag{40}$$

and in particular implies that $n \geq 4$ (see Appendix H for the proof of this implication).

I now show that (40) implies (39) by showing that for $n \geq 4$ we have:

$$\big(1 - (1-q)^n\big)\big[n(1-q) - (1-q^n)\big] > n(1-q)^2 q.$$

Note that $\big(1 - (1-q)^n\big) > \big(1 - (1-q)^2\big) = q(2-q)$ for $n \geq 4$, hence it suffices to show

$$q(2-q)\big[n(1-q) - (1-q^n)\big] > n(1-q)^2 q$$
$$\Leftrightarrow (2-q)n(1-q) - n(1-q)^2 > (2-q)(1-q^n)$$
$$\Leftrightarrow n(1-q) > (2-q)(1-q^n)$$
$$\Leftrightarrow n(1-q) > (2-q)(1-q)\sum_{k=0}^{n-1} q^k$$
$$\Leftrightarrow n > (2-q)\sum_{k=0}^{n-1} q^k = (1-q)\sum_{k=0}^{n-1} q^k + \sum_{k=0}^{n-1} q^k = 1 - q^n + \sum_{k=0}^{n-1} q^k.$$

Consider the function $f(q) \equiv 1 - q^n + \sum_{k=0}^{n-1} q^k$. Differentiating it, we get:

$$f'(q) = -nq^{n-1} + \sum_{k=1}^{n-1} kq^{k-1} > -nq^{n-1} + \sum_{k=1}^{n-1} kq^{n-1} = q^{n-1}\left[\sum_{k=1}^{n-1} k - n\right] = q^{n-1} n \frac{(n-3)}{2},$$

which is strictly positive since $n \geq 4$. We can conclude that $f(q)$ is strictly increasing on $(0,1)$, moreover $f(1) = 1 - 1^n + \sum_{k=0}^{n-1} 1^k = n$, therefore $f(q) < n$ for all $q \in (0,1)$. □

G Proofs of Lemmas 9, 12, 15

(Full-surplus-extracting collusive public perfect equilibria)

G.1 Proof of Lemma 9

Proof. Full surplus extraction and $\mathcal{R}^*_{\text{fse}}(\delta) \geq (1-q^n)\overline{\theta}$ are shown in the main text, (Eq-payoff), (LowIC) and (Col-sep-1) are satisfied by construction, and (HighIC-up) is also checked in the main text, hence it remains to check the incentive constraints (HighIC-down) and (HighIC-on-sch), and the collusiveness constraints (Col-sep-2) and (Col-pool).

Incentive constraints. Let us start with (HighIC-down). Evaluated at $\left(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}\right)$, it is:

$$\text{(HighIC-down)} \qquad (1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*) + \delta v^*_{\text{fse}} \geq (1-\delta)q^{n-1}(\overline{\theta}-\underline{b}^*).$$

Plugging $\overline{b}^*$, $\underline{b}^*$, and v^*_{fse} in, we obtain:

$$\frac{(1-\delta)(1-q^n)}{n(1-q)}\frac{q^n(1-\delta(1-q))(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} + \frac{\delta}{n}\frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)}$$
$$\geq (1-\delta)q^{n-1}\frac{q^n\big(1-\delta(1-q)^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)},$$

which simplifies to:

$$\begin{aligned} &\frac{1-q^n}{n(1-q)}(1-\delta(1-q)) + \frac{\delta}{n}(1-q^n) \geq q^{n-1}\big(1-\delta(1-q)^n\big) \\ \Leftrightarrow\ &\frac{1-q^n}{n(1-q)} - q^{n-1} \geq -q^{n-1}\delta(1-q)^n \end{aligned} \tag{41}$$

which is true since the left-hand side of (41) is strictly positive.

Consider now (HighIC-on-sch). Evaluated at $\left(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}\right)$, it is:

$$\text{(HighIC-on-sch)} \qquad \frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*) \geq \frac{q^{n-1}}{n}(\overline{\theta}-\underline{b}^*).$$

Plugging $\overline{b}^*$ and $\underline{b}^*$ in, we obtain:

$$\frac{1-q^n}{1-q}\frac{q^n(1-\delta(1-q))(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} \geq q^{n-1}\frac{q^n\big(1-\delta(1-q)^n\big)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)},$$

which simplifies to:

$$\begin{aligned}
&\frac{1-q^n}{1-q}\big(1-\delta(1-q)\big) \geq q^{n-1}\big(1-\delta(1-q)^n\big)\\
\Leftrightarrow\ &\frac{1-q^n}{1-q} - \delta(1-q^n) \geq q^{n-1} - q^{n-1}\delta(1-q)^n\\
\Leftrightarrow\ &\frac{(1-q)\sum_{k=0}^{n-1} q^k}{1-q} - \delta(1-q^n) \geq q^{n-1} - q^{n-1}\delta(1-q)^n\\
\Leftrightarrow\ &\frac{1}{\delta}\sum_{k=0}^{n-2} q^k \geq (1-q^n) - q^{n-1}(1-q)^n.
\end{aligned}$$

Since $\frac{1}{\delta}\sum_{k=0}^{n-2} q^k > \sum_{k=0}^{n-2} q^k$, it is enough to show that $\sum_{k=0}^{n-2} q^k \geq (1-q^n) - q^{n-1}(1-q)^n$, which simplifies to $\sum_{k=1}^{n-2} q^k + q^n \geq -q^{n-1}(1-q)^n$, which is clearly true.

Collusiveness constraints. Consider (Col-sep-2) first. Evaluated at $\big(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}\big)$, it is:

$$\text{(Col-sep-2)} \qquad v^*_{\text{fse}} \geq \frac{(1-\delta)\big[(1-q^n)(\overline{\theta}-\underline{b}^*) + q^n(\underline{\theta}-\underline{b}^*)\big]}{n(1-\delta q^n)}.$$

Plugging $\overline{b}^*$, $\underline{b}^*$, and v^*_{fse} in, we obtain:

$$\frac{1}{n}\frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)} \geq \frac{(1-\delta)q^n(1-q^n)\big(1-\delta(1-q)^n-\delta q\big)(\overline{\theta}-\underline{\theta})}{n(1-\delta q^n)\big(\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)\big)},$$

which simplifies to $1 \geq \frac{1-\delta(1-q)^n-\delta q}{1-\delta q^n}$, which in turn simplifies to $(1-q)^n \geq -q + q^n$, which is clearly true.

Consider (Col-pool) now. Evaluated at $\big(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}\big)$, it is:

$$\text{(Col-pool)} \qquad v^*_{\text{fse}} \geq \frac{1}{n}\big[(1-q)(\overline{\theta}-\underline{b}^*) + q(\underline{\theta}-\underline{b}^*)\big].$$

Plugging $\overline{b}^*$ and $\underline{b}^*$ in, we obtain:

$$v^*_{\text{fse}} \geq \frac{\big((1-q)q^n\big(1-\delta(1-q)^n\big) - \delta q^2(1-q^n)\big)(\overline{\theta}-\underline{\theta})}{n\big(\delta q\big(1-q^n\big)+q^n\big(1-\delta(1-q)^n\big)\big)}. \tag{42}$$

Consider the numerator of the right-hand side of (42) in the limit as δ goes to 1:

$$\begin{aligned}
(1-q)q^n\big(1-(1-q)^n\big) - q^2(1-q^n) &= (1-q)\left[q^n\big(1-(1-q)^n\big) - q^2\sum_{k=0}^{n-1} q^k\right]\\
&= (1-q)\left[q^n - q^n(1-q)^n - q^2\sum_{k=0}^{n-3} q^k - q^n - q^{n+1}\right]\\
&= (1-q)\left[-q^n(1-q)^n - q^2\sum_{k=0}^{n-3} q^k - q^{n+1}\right] < 0.
\end{aligned}$$

Recall that v^*_{fse} is positive, whereas the right-hand side of (42) goes to a negative value. By continuity, there is a $\delta^* \in (0,1)$ such that for all $\delta > \delta^*$ (Col-pool) is satisfied.

□

G.2 Proof of Lemma 12

Proof. Full surplus extraction and $\mathcal{R}^*_{\text{fse}}(\delta) \geq (1-q^n)\overline{\theta}$ are shown in the main text, (Eq-payoff), (HighIC-up) and (Col-sep-1) are satisfied by construction, (LowIC) and (HighIC-down) are also checked in the main text, hence it remains to check the incentive constraint (HighIC-on-sch), and the collusiveness constraints (Col-sep-2) and (Col-pool).

Incentive constraint. Let us start with (HighIC-on-sch). Evaluated at $\left(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}\right)$ it is:

$$\text{(HighIC-on-sch)} \qquad \frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*) \geq \frac{q^{n-1}}{n}(\overline{\theta}-\underline{b}^*).$$

Plugging $\overline{b}^*$ and $\underline{b}^*$, we obtain:

$$\frac{1-q^n}{n(1-q)}\frac{1}{D(\delta)}\delta q^n(1-q^n)(1-q)(\overline{\theta}-\underline{\theta}) \geq \frac{q^{n-1}}{n}\frac{1}{D(\delta)}q^n\left(1-\delta(1-q)^n\right)\left[n(1-q)-(1-q^n)\right](\overline{\theta}-\underline{\theta}),$$

which is equivalent to:

$$\delta(1-q^n)(1-q^n) \geq q^{n-1}\left(1-\delta(1-q)^n\right)\left[n(1-q)-(1-q^n)\right],$$

which is in particular true whenever

$$\delta(1-q^n)(1-q^n) \geq q^{n-1}\left[n(1-q)-(1-q^n)\right],$$

i.e. for all δ satisfying $\delta \geq \frac{q^{n-1}\left[n(1-q)-(1-q^n)\right]}{(1-q^n)(1-q^n)}$. Note that such δ exist in (0,1) since

$$\begin{aligned}
&(1-q^n)(1-q^n) > q^{n-1}\left[n(1-q)-(1-q^n)\right] \qquad (43)\\
&\Leftrightarrow (1-q^n)(1-q^n) + q^{n-1}(1-q^n) > nq^{n-1}(1-q)\\
&\Leftrightarrow (1-q^n)(1+q^{n-1}(1-q)) > nq^{n-1}(1-q)\\
&\Leftrightarrow (1+q^{n-1}(1-q))\sum_{k=0}^{n-1} q^k > nq^{n-1},
\end{aligned}$$

where the last inequality is true since $\sum_{k=0}^{n-1} q^k > nq^{n-1}$ and $1+q^{n-1}(1-q) > 1$. Thus the high type on-schedule incentive compatibility constraint is satisfied for a sufficiently high δ.

Collusiveness constraints. Consider (Col-sep-2) first. Evaluated at $\left(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}\right)$, it is

$$\text{(Col-sep-2)} \qquad v^*_{\text{fse}} \geq v(\underline{b}^*+\epsilon, \underline{b}^*) = \frac{(1-\delta)\left[(1-q^n)(\overline{\theta}-\overline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\right]}{n(1-\delta q^n)}.$$

Plugging $\overline{b}^*$, $\underline{b}^*$ and v^*_{fse} in, we obtain:

$$\frac{1-\delta}{nD(\delta)}q^n(1-q^n)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}) \geq$$
$$\geq \frac{(1-\delta)q^n(1-q^n)(\overline{\theta}-\underline{\theta})}{n(1-\delta q^n)D(\delta)}\bigg(\big(1-\delta(1-q)^n\big)\big[n(1-q)-(1-q^n)\big] - \big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big]\bigg),$$

which simplifies to:

$$\big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big] \geq \delta\big(q^n-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big],$$

which can only be satisfied if:

$$\delta \geq \frac{n(1-q)-(1-q^n)}{n(1-q)-q(1-q^n)-\big(q^n-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]}. \tag{44}$$

Such values of δ exist in $(0,1)$ only if the right-hand side of (44) is strictly below 1, i.e. when

$$(1-q^n)(1-q) > \big(q^n-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]. \tag{45}$$

It is easy to show that the above inequality is implied by the parameter restrictions of Case 2. Those restrictions in particular imply that:

$$(1-q^n)(1-q) > q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big].$$

Observe that $q^n-(1-q)^n < q^{n-1}\big(1-(1-q)^n\big)$, which establishes (45).

Consider (Col-pool) now. Evaluated at $\big(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}\big)$, it is

$$\text{(Col-pool)} \qquad v^*_{\text{fse}} \geq \frac{1}{n}\big[(1-q)(\overline{\theta}-\underline{b}^*)+q(\underline{\theta}-\underline{b}^*)\big].$$

Plugging $\overline{b}^*$ and $\underline{b}^*$ in, we obtain

$$v^*_{\text{fse}} \geq \frac{1}{n}\bigg[(1-q)\frac{1}{D(\delta)}q^n\big(1-\delta(1-q)^n\big)\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}) - q\frac{1}{D(\delta)}\big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big](1-q^n)(\overline{\theta}-\underline{\theta})\bigg]. \tag{46}$$

I show that the right-hand side of (46) is strictly negative in the limit as δ goes to 1. In the limit it is given by:

$$\frac{(1-q)(\overline{\theta}-\underline{\theta})}{nD(1)}\big[q^n\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]-q(1-q^n)(1-q^n)\big]. \tag{47}$$

Observe that the expression in (47) is strictly negative if and only if

$$(1-q^n)(1-q^n) > q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big],$$

which is true since $(1-q^n)(1-q^n) > q^{n-1}\big[n(1-q)-(1-q^n)\big]$, as established by (43). Recall that v^*_{fse} is positive, whereas the right-hand side of (46) goes to a negative value. By continuity, there is a $\delta^* \in (0,1)$ such that for all $\delta > \delta^*$ (Col-pool) is satisfied.

□

G.3 Proof of Lemma 15

Proof. Full surplus extraction and $\mathcal{R}^*_{\text{fse}}(\delta) \geq (1-q^n)\overline{\theta}$ are shown in the main text, (Eq-payoff), (HighIC-up) and (HighIC-down) are satisfied by construction, (LowIC) and (Col-sep-1) are also checked in the main text, hence it remains to check the incentive constraint (HighIC-on-sch), and the collusiveness constraints (Col-sep-2) and (Col-pool).

Incentive constraint. Let us start with (HighIC-on-sch). Evaluated at $\big(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}\big)$, it is:

$$\text{(HighIC-on-sch)} \qquad \frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*) \geq \frac{q^{n-1}}{n}(\overline{\theta}-\underline{b}^*).$$

Note that both $\overline{\theta}-\overline{b}^*$ and $\overline{\theta}-\underline{b}^*$ are strictly positive for δ high enough. Observe that we have $\overline{\theta}-\overline{b}^* = q^{n-1}(\overline{\theta}-\underline{b}^*)$ in Case 3 by construction, and therefore:

$$\frac{1-q^n}{n(1-q)}(\overline{\theta}-\overline{b}^*) > \frac{1}{n}(\overline{\theta}-\overline{b}^*) = \frac{q^{n-1}}{n}(\overline{\theta}-\underline{b}^*).$$

where the first inequality is true since $1-q^n > 1-q$ for $n \geq 2$ and $q \in (0,1)$, implying that the high-type on-schedule incentive compatibility is satisfied.

Collusiveness constraints. Consider (Col-sep-2) first. Evaluated at $\big(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}\big)$, it is:

$$\text{(Col-sep-2)} \qquad v^*_{\text{fse}} \geq v(\underline{b}^*+\epsilon, \underline{b}^*) = \frac{(1-\delta)\big[(1-q^n)(\overline{\theta}-\underline{b}^*)+q^n(\underline{\theta}-\underline{b}^*)\big]}{n\big(1-\delta q^n\big)}.$$

Plugging $\overline{b}^*$, $\underline{b}^*$, and v^*_{fse} in, we obtain:

$$\frac{1}{nD(\delta)}(1-\delta)q^n\big[n(1-q)-(1-q^n)\big](\overline{\theta}-\underline{\theta}) \tag{48}$$
$$\geq \frac{(1-\delta)(\overline{\theta}-\underline{\theta})}{nD(\delta)\big(1-\delta q^n\big)}\Big[(1-q^n)\delta q(1-q) - q^n\big[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\big]\Big],$$

which is equivalent to:

$$q^n\left[n(1-q)-(1-q^n)\right] \\ \geq \frac{1}{(1-\delta q^n)}\left[(1-q^n)\delta q(1-q)-q^n\left[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\right]\right],$$

which holds for high enough δ whenever it holds as a strict inequality at $\delta = 1$, i.e. whenever

$$q^n\left[n(1-q)-(1-q^n)\right] > \frac{1}{(1-q^n)}\left[(1-q^n)q(1-q)-q^n(1-q^n)(1-q)\right]$$
$$\Leftrightarrow q^{n-1}\left[n(1-q)-(1-q^n)\right] > (1-q)(1-q^{n-1}). \tag{49}$$

The inequality in (49) is true since:

$$(1-q)(1-q^{n-1}) < (1-q)(1-q^n) \leq q^{n-1}\left(1-(1-q)^n\right)\left[n(1-q)-(1-q^n)\right],$$

where the second inequality is true in Case 3 by assumption.

Consider (Col-pool). Evaluated at $\left(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}\right)$, it is

$$\text{(Col-pool)} \qquad v^*_{\text{fse}} \geq \frac{1}{n}\left[(1-q)(\overline{\theta}-\underline{b}^*)+q(\underline{\theta}-\underline{b}^*)\right].$$

Plugging $\overline{b}^*$ and $\underline{b}^*$ in, we obtain:

$$v^*_{\text{fse}} \geq \frac{\overline{\theta}-\underline{\theta}}{nD(\delta)}\left[(1-q)\delta q(1-q)-q\left[(1-q^n)(1-\delta q)-n(1-\delta)(1-q)\right]\right]. \tag{50}$$

As in the previous two cases, I show that the right-hand side of (50) goes to a negative value as δ goes to 1. Indeed the limit of the right-hand side of (50) is given by:

$$\frac{\overline{\theta}-\underline{\theta}}{nD(1)}\left[(1-q)q(1-q)-q(1-q^n)(1-q)\right] = \frac{q(1-q)(\overline{\theta}-\underline{\theta})}{nD(1)}\left[q^n-q\right] < 0.$$

Recall that v^*_{fse} is positive, hence, by continuity, there is a $\delta^* \in (0,1)$ such that for all $\delta > \delta^*$ (Col-pool) is satisfied.

□

H Some results on the parameter regions

Case 1: High expected valuation/Small number of buyers

The range of parameters, where Case 1 applies, is given by $q < \frac{1-q^n}{n(1-q)}$. It is easy to check that this condition can be satisfied for any q as long as $n = 2$ or $n = 3$, but only for some q if $n \geq 4$. Indeed, consider $n = 2$ first. In this case the condition becomes:

$$2q < \frac{1-q^2}{1-q} \Leftrightarrow 2q < 1 + q \Leftrightarrow q < 1,$$

which is obviously true. If $n = 3$, the condition becomes:

$$3q < \frac{1-q^3}{1-q} \Leftrightarrow 3q < 1 + q + q^2 \Leftrightarrow 0 < 1 - 2q + q^2 \Leftrightarrow 0 < (1-q)^2,$$

which is also obviously true for any $q \in (0, 1)$. If $n = 4$, the condition becomes:

$$\begin{aligned} 4q < \frac{1-q^4}{1-q} &\Leftrightarrow 4q < 1 + q + q^2 + q^3 \Leftrightarrow 0 < 1 - 3q + q^2 + q^3 \\ &\Leftrightarrow 0 < (1-q)(-q^2 - 2q + 1) \Leftrightarrow 0 < -q^2 - 2q + 1, \end{aligned}$$

which is only true for $q \in (0, -1 + \sqrt{2})$. It is possible to establish that for any number of players $n \geq 4$ there will be some values of q falling into Case 1:

Proposition 3. *The equation* $1 - q^n = nq(1-q)$ *has a unique solution* q^* *on* $(0, 1)$ *for any* $n \geq 4$. *Moreover for all* $q < q^*$ *it is true that* $q < \frac{1-q^n}{n(1-q)}$ *and vice versa.*

Proof. Both sides of the equation can be divided by $1 - q$ to obtain: $\sum_{k=0}^{n-1} q^k - nq = 0$, which can again be divided by $1 - q$ to obtain: $1 - \sum_{k=1}^{n-2}(n-1-k)q^k = 0$. Define the function:

$$g(q) \equiv 1 - \sum_{k=1}^{n-2}(n-1-k)q^k.$$

Clearly $g(0) = 1$, and $g(1)$ is given by:

$$\begin{aligned} g(1) &= 1 - \sum_{k=1}^{n-2}(n-1-k) = 1 - (n-1)(n-2) + \sum_{k=1}^{n-2} k \\ &= 1 - (n-1)(n-2) + \frac{(n-1)(n-2)}{2} = 1 - \frac{(n-1)(n-2)}{2} = \frac{n}{2}(3-n) < 0. \end{aligned}$$

hence the equation has a solution on $(0, 1)$ for every $n \geq 4$ by the Intermediate Value Theorem.

Consider now the derivative of $g(\cdot)$:

$$g'(q) = -\sum_{k=1}^{n-2}(n-1-k)kq^{k-1} < 0,$$

which implies that the solution q^* is unique and that $q < \frac{1-q^n}{n(1-q)}$ for all $q < q^*$ and vice versa. □

The above proposition essentially shows that for every $n \geq 4$ the restriction divides the interval $(0,1)$ into two parts. In the left part of the segment one will find the values of q that fall into Case 1, and in the right part of the segment one will find the values of q that fall into Cases 2 and 3. Figure 3 provides an illustration and also suggests that, as n goes to infinity, lower and lower values of q fall into Case 1 until there are none left in the limit. Indeed, it is easy to see that $\lim_{n\to\infty} nq(1-q)-(1-q^n) = +\infty$, implying that, for any fixed value of q, the parameter restriction does not hold for all sufficiently high n.

Case 2: Medium expected valuation

The parameter restrictions of Case 2 in particular imply that:

$$(1-q^n)(1-q) > q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]. \tag{51}$$

In the following proposition I establish that the set of q satisfying (51) is non-empty for any $n \geq 4$ and that there are values q that do not satisfy (51) for every $n \geq 4$.

Proposition 4. *The equation*

$$(1-q^n)(1-q) = q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]$$

has a solution on $(0,1)$ *for every* $n \geq 4$.

Proof. Consider the equation:

$$\begin{aligned}
&(1-q^n)(1-q) = q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]\\
&\Leftrightarrow (1-q^n) = q^{n-1}\big(1-(1-q)^n\big)\left[n-\sum_{k=0}^{n-1}q^k\right]\\
&\Leftrightarrow (1-q)\sum_{k=0}^{n-1}q^k = q^{n-1}\big(1-(1-q)^n\big)(1-q)\sum_{k=0}^{n-2}(n-1-k)q^k\\
&\Leftrightarrow \sum_{k=0}^{n-1}q^k = q^{n-1}\big(1-(1-q)^n\big)\sum_{k=0}^{n-2}(n-1-k)q^k.
\end{aligned}$$

and consider the function:

$$g(q) = q^{n-1}\big(1-(1-q)^n\big)\sum_{k=0}^{n-2}(n-1-k)q^k - \sum_{k=0}^{n-1} q^k.$$

Clearly $g(0) = -1$ and $g(1)$ is computed as:

$$\begin{aligned} g(1) &= \sum_{k=0}^{n-2}(n-1-k)1^k - \sum_{k=0}^{n-1} 1^k \\ &= (n-1)^2 - \sum_{k=0}^{n-2} k - n \\ &= (n-1)^2 - \frac{(n-1)(n-2)}{2} - n = n\frac{n-3}{2} > 0. \end{aligned}$$

The result follows by continuity of $g(q)$. □

Recall from Figure 3 that the range of q falling into Case 2 expands as n increases. In the next proposition I establish that any $q \in (0,1)$ will satisfy condition (51) for all sufficiently high values of n:

Proposition 5. *For all* $q \in (0,1)$

$$\lim_{n\to\infty}\Big((1-q^n)(1-q) - q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]\Big) = 1-q > 0.$$

Proof. Note that the expression can be rewritten as:

$$\underbrace{(1-q^n)(1-q)}_{\to 1-q \text{ as } n\to\infty} - nq^{n-1}\underbrace{(1-q)\big(1-(1-q)^n\big)}_{\to 1-q \text{ as } n\to\infty} + \underbrace{q^{n-1}(1-q^n)\big(1-(1-q)^n\big)}_{\to 0 \text{ as } n\to\infty}.$$

It thus remains to check that $\lim_{n\to\infty} nq^{n-1} = 0$. Taking logs, I get:

$$\begin{aligned} \log\left(nq^{n-1}\right) = \log(n) + (n-1)\log(q) &\le \sqrt{n-1} + (n-1)\log(q) \\ &= (n-1)\left(\frac{1}{\sqrt{n-1}} + \log(q)\right). \end{aligned}$$

Note that since $\log(q)$ is strictly negative and $\frac{1}{\sqrt{n-1}}$ goes to 0 as n goes to infinity, we have for a large enough n:

$$(n-1)\left(\frac{1}{\sqrt{n-1}} + \log(q)\right) \le (n-1)\frac{\log(q)}{2}.$$

Since $\log(q) < 0$ we have $\lim_{n\to\infty}(n-1)\frac{\log(q)}{2} = -\infty$, but then $\lim_{n\to\infty}\log\left(nq^{n-1}\right) = -\infty$, which establishes the claim. □

Figure 3 also suggests that the restriction in (51) can be satisfied for all $q \leq \frac{1}{2}$. Indeed, this claim can be shown formally:

Proposition 6. *For all $q \in (0, \frac{1}{2}]$ it is true that*

$$(1-q^n)(1-q) > q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big].$$

Proof. The parameter restriction can be rewritten as:

$$\frac{1-q^n}{1-(1-q)^n} > q^{n-1}\left[n - \sum_{k=0}^{n-1} q^k\right].$$

Observe that $\frac{1-q^n}{1-(1-q)^n} \geq 1$ for all $q \leq \frac{1}{2}$ since $1-q^n \geq 1-(1-q)^n$ is equivalent to $1-q \geq q$. It thus suffices to show that $1 \geq nq^{n-1}$ for all $q \in (0, \frac{1}{2}]$. Define the function $f(q) = nq^{n-1}-1$. It is clearly strictly increasing in q since $f'(q) = n(n-1)q^{n-2}$. It thus suffices to check that the claim is true for $q = \frac{1}{2}$ or $1 \geq n\frac{1}{2^{n-1}}$ which is equivalent to $2^{n-1} \geq n$, which is true for all $n \geq 2$. □

Case 3: Low expected valuation

The range of parameters, where Case 3 applies, is defined by:

$$q \geq 1 - \frac{q^{n-1}\big(1-(1-q)^n\big)\big[n(1-q)-(1-q^n)\big]}{1-q^n}.$$

Recall that it in particular implies that $q \geq \frac{1-q^n}{n(1-q)}$ in Case 3. Recall also that $q \geq \frac{1-q^n}{n(1-q)}$ implies that $n \geq 4$ because it cannot be satisfied for any q as long as $n = 2$ or $n = 3$. Combined with the result of Proposition 4, it implies that Case 3 applies to some values of q for all $n \geq 4$, and does not apply to any values of q for $n = 2$ or $n = 3$ (see Figure 3 for an illustration).

References

Abdulkadiroglu, A. and K. Chung (2004), "Auction design with tacit collusion." Working paper.

Abreu, D., D. Pearce, and E. Stacchetti (1990), "Toward a theory of discounted repeated games with imperfect monitoring." *Econometrica*, 58 (5), 1041–1063.

Athey, S., K. Bagwell, and C. Sanchirico (2004), "Collusion and price rigidity." *Review of Economic Studies*, 71 (2), 317–349.

Bergemann, D. and J. Hörner (2018), "Should first-price auctions be transparent?" *American Economic Journal: Microeconomics*, 10 (3), 177–218.

Bergemann, D. and J. Välimäki (2019), "Dynamic mechanism design: An introduction." *Journal of Economic Literature*, 57 (2), 235–274.

Blackwell, D. (1965), "Discounted dynamic programming." *The Annals of Mathematical Statistics*, 36 (1), 226–235.

Carroll, G. (2019), "Robustness in mechanism design and contracting." *Annual Review of Economics*, 11, 139–166.

Chassang, S., K. Kawai, J. Nakabayashi, and J. Ortner (2022), "Robust screens for non-competitive bidding in procurement auctions." *Econometrica*, 90 (1), 315–346.

Che, Y.K. and J. Kim (2009), "Optimal collusion-proof auctions." *Journal of Economic Theory*, 144 (2), 565–603.

Correia-da-Silva, J. (2017), "A survey on the theory of collusion under adverse selection in auctions and oligopolies." Working paper.

Cramton, P., R. Gibbons, and P. Klemperer (1987), "Dissolving a partnership efficiently." *Econometrica*, 55, 615–632.

Fudenberg, D., D. Levine, and E. Maskin (1994), "The folk theorem with imperfect public information." *Econometrica*, 62, 997–1040.

Iossa, E., S. Loertscher, L. M. Marx, and P. Rey (2023), "Coordination in the fight against collusion." *American Economic Journal: Microeconomics*. Forthcoming.

Kandori, M. and I. Obara (2006), "Towards a belief-based theory of repeated games with private monitoring, an application of POMDP." Working paper.

Luenberger, D. G. and Y. Ye (2021), *Linear and Nonlinear Programming*. Springer Cham.

Mailath, G. J. and L. Samuelson (2006), *Repeated Games and Reputations: Long-run relationships*. Oxford University Press, New York.

McAfee, R. P. and J. McMillan (1992), "Bidding rings." *American Economic Review*, 82 (3), 579–599.

Ortner, J., S. Chassang, J. Nakabayashi, and K. Kawai (2022), "Screening adaptive cartels." Working paper.

Pavan, A., I. Segal, and J. Toikka (2014), "Dynamic mechanism design: a Myersonian approach." *Econometrica*, 82 (2), 601–653.

Skrzypacz, A. and H. Hopenhayn (2004), "Tacit collusion in repeated auctions." *Journal of Economic Theory*, 114 (1), 153–169.

Thomas, C. J. (2005), "Using reserve prices to deter collusion in procurement competition." *Journal of Industrial Economics*, 53 (3), 301–326.

Zhang, W. (2022), "Collusion enforcement in repeated first-price auctions." *Theoretical Economics*, 17 (4), 1847–1895.